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Equiconvergence theorems for differential operators

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Abstract

The paper is a survey dedicated to topic in the title. In chapter 1 we expose the most advanced equiconvergence results for Birkhoff- or Stone-regular differential operators. Considerable part of them was obtained by the Saratov mathematical school but is published in the literature that is hard to come by.

We present also an author's (commutator) approach to equiconvergence and derive a stonger form of the Riemann localization principle as well as a first equiconvergence result (not equisummability) for multidimensional Schrödinger operator.

Chapter 2 contains a full proof of the equiconvergence on the whole interval, which constitutes a true generalization of the Tamarkin-Stone theorem.

Given Birkhoff-regular ordinary differential operator L in $L^2(0,1)$ and continuous function f, which belongs to closure of D_L in $C[0,1]$, we establish necessary and sufficient conditions for uniform equiconvergence on $[0,1]$ of the eigenfunction expansion of L and of trigonometric Fourier integral of the modified function

$$\tilde{f}(x) = \begin{cases} f(0), & x < 0; \\ f(x), & 0 \leq x \leq 1; \\ f(1), & x > 1. \end{cases}$$

These conditions consist of uniform converge to 0 on $[0,\delta]$ for some (any) $0 < \delta < 1$ of certain singular integrals acting upon specified linear combinations of functions $f(x)$ and $f(1-x)$.

Chapters 3 and 4 apply our approach to equiconvergence to singular self-adjoint differential operators, generalizing well-known results of A.G.Kostuchenko, and to general serii in eigenfunctions.

Preface

Many central problems of the spectral theory of linear operators are concentrated around the problem of eigenfunction expansions. From one hand it accumulates questions of eigenvalues and eigenfunctions asymptotics, from the other it connects mathematics with many physical problems of string and membrane vibrations, of quantum mechanics and so on. There are two most elaborated parts of this theory, firstly, the spectral theory of self-adjoint operators (ordinary, singular and in partial derivatives) and secondly, that of boundary value problems in a finite interval.

Herewith already at the beginning of the 20th century G.D.Birkhoff discovered an important class of *regular* higher order boundary value problems. Just immediately J.Tamarkin observed that for two such problems the difference of eigenfunction expansions converges to zero in any interior point of the main interval. This phenomenon was called *equiconvergence* and it makes possible to reduce numerous questions of point and uniform convergence to those of some model, usually, trigonometric system.

This remarquable result has several predecessors in the case of second order operators. We shall mention V.A.Steklov, E.W.Hobson and A.Haar. Later on it was generalized by J.Tamarkin and by M.Stone and yielded a large field of investigations with a lot of off-shoots and generalizations. In the present review we set ourselves a task of exposing the current state-of-arts in this domain with a strong emphasis to describe the most advanced achievements, at least to the best of our knowledge. During the past two decades the author himself developped new approaches to these questions and obtained solutions of several long standing problems. However, they are published in a literature which is hard to come by. Partially they remain yet unpublished or appeared only in conferences proceedings. Therefore we place here the most important of them with complete demonstrations. In particular, a solution is given to the problem of *equiconvergence on the whole interval* obtained in 1992. We also present a complete investigation of higher order singular self-adjoint quasidifferential operators under minimal possible restrictions.

History of the question is exposed but with no claim of completeness. Hence, the bibliography is rather large but not exhaustive.

During the exposition we formulate several conjectures reflecting our own understanding of the subject in order to stimulate further investigations whether these conjectures will happen to be true or not.

Due to the lack of time as well as author's insufficient knowledge we omit some important topics, for instance, opearator bundles, differential expressions with multiple roots of the characteristic equation and all the more with varying multiplicity roots (equations with a turning point), operators in partial derivatives. We only touch the latter once. These questions deserve separate review or reviews.

During the preparation we have fruitfull discussions with our colleagues. They also provided us many materials used in the paper. In particular, Professor A.P.Khromov has kindly given us permission to use his unpublished notes on the problem of equiconvergence.

Therefore we take an opportunity to thank A.P.Khromov, B.É.Kunyavskii, S.N.Kuptsov, G.V.Radzievskii, V.S.Rykhlov and I.Yu.Trushin though the list may have been considerably increased.

Throughout the paper the reader will often meet citations of N.P.Kuptsov's [1925-1995] results. However, equiconvergence reflects only one of the numerous fields of interest and activity of this universal mathematician. He published very selectively but his impact on the developement of the Saratov school on spectral theory is difficult to overestimate. Therefore we dared to dedicate him this review as a small tribute to his memory.

In the body of the text we use some notations which has become standard.

- b.v.p. — boundary value problems,
- e.f. — eigenfunctions,
- e.a.f. — eigenfunctions and associated eigenfunctions,
- e.v. — eigenvalues,
- ch.v. — characteristic values,
- f.s.s. — fundamental system of solutions,
- sp.f. — spectral function,
- g.sp.f. — generalized spectral function,
- s.a. — self-adjoint,
- span — minimal closed subspace, containing a given set of elements,
- $[a] := a + O(1/\rho)$ — the Birkhoff's symbol.
- $Entier(h)$ — the largest integer $\leq h$.

Chapter 1

Introduction

To the memory of N.P.Kuptsov

1 Birkhoff-regular problems

1.1 Early results

Let us consider a differential operator L in $L^2(0,1)$ defined by a two-point b.v.p. (($D = -id/dx$)):

$$l(y) \equiv D^n y + \sum_{k=0}^{n-2} p_k(x) D^k y = \lambda y, \quad 0 \le x \le 1, \quad p_k \in L(0,1) \tag{1.1}$$

and n linearly independent normalized boundary conditions [97, p.65–66]:

$$\begin{gathered} U_j(y) \equiv V_j(y) + \ldots = 0, \quad j = 0, \ldots, n-1, \\ V_j(y) \equiv b_j^0 D^j y(0) + b_j^1 D^j y^(1). \end{gathered} \tag{1.2}$$

Here the ellipsis takes place of lower order terms at 0 and at 1. Further b_j^0, b_j^1 are column vectors of length r_j, where

$$0 \le r_j \le 2, \ \sum_{k=0}^{n-1} r_j = n, \quad rank[b_j^0 b_j^1] = r_j.$$

This form of normalized boundary conditions was first introduced by S.Salaff [109, p.356–357]. It is evident that $r_j = 0$ implies the absence of order j conditions. In the case $r_j = 2$ we merely put

$$[b_j^0 b_j^1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

At the beginning of the 20th century G.D.Birkhoff discovered a famous broad class of b.v.p. with remarkable spectral properties [11, 12]. Recall its definition.

Definition 1.1. Let $q = Entier(n/2)$, $\varepsilon_j := \exp(2\pi ij/n)$, $k = 0, .., n-1$;

$$b^i = (b^i_j)^{n-1}_{j=0} = \begin{bmatrix} b^i_0 \\ \vdots \\ b^i_{n-1} \end{bmatrix}, \ \mathbf{B}^i_k = [b^i_j \cdot \varepsilon^j_k]^{n-1}_{j=0}, \ i = 0, 1; \tag{1.3}$$

$$\theta(b^0, b^1, L) = \det[\mathbf{B}^0_k, k = 0, \ldots, q-1 | \mathbf{B}^1_k, k = q, \ldots, n-1] \tag{1.4}$$

The vertical line | separates columns with superscripts 0 and 1. We shall call boundary conditions (1.2) and the corresponding operator L **Birkhoff-regular** and write $L \in (R)$ if

$$\begin{cases} \theta(b^0, b^1, L) \neq 0, & n = 2q, \\ \theta(b^0, b^1, L) \neq 0 \text{ and } \theta(b^1, b^0, L) \neq 0, & n = 2q+1. \end{cases} \tag{1.5}$$

This form of Birkhoff-regularity was invented by S.Salaff[109, p.361] who has done a first serious investigation of the nature of the regularity determinants.

It is worth noting here that recently it was shown [91] that

Theorem 1.2 (A.M.Minkin). $L \in (R) \Longleftrightarrow L^2 \in (R)$.

This property serves to reduce odd order problems to the even ones without any separate treatment of the former.

Let $\{\lambda_j\}^\infty_1$ be the set of all e.v. of L, $G(x, \xi, \varrho)$ be its Green function, i.e. the resolvent's $R_\lambda := (L - \lambda I)^{-1}$ kernel,

$$\varrho = \lambda^{1/n}, \qquad |\varrho| = |\lambda|^{1/n},$$

where

$$\arg \varrho = \arg \lambda / n, \qquad 0 \leq \arg \lambda \leq 2\pi, \tag{1.6}$$

if n is even and

$$\begin{cases} \arg \varrho = \arg \lambda / n, & -\pi/2 \leq \arg \lambda \leq \pi/2; \\ \arg \varrho = \pi - \frac{\pi - \arg \lambda}{n}, & \pi/2 \leq |\arg \lambda| \leq \pi \end{cases}$$

if it is odd. Then

$$\varrho \in S_0 = S_1 \cup S_2, \quad S_k = \left\{ (k-1)\frac{\pi}{n} \leq \arg \varrho < \frac{k\pi}{n} \right\}, \qquad n \text{ even} \tag{1.7}$$

$$\varrho \in S_0 = S_1 \cup S_2 = \{ |\arg \varrho| \leq \pi/2n \} \cup \{ |\pi - \arg \varrho| \leq \pi/2n \}, \qquad n \text{ odd}. \tag{1.8}$$

Ch.v. $\varrho_j = \lambda^{1/n}_j$ of the operator $L \in (R)$ tend asymptotically to one arithmetic progression if n is odd and to two ones if it is even:

$$
\begin{aligned}
\varrho_j &= 2\pi j + c + o(1), \quad n = 2q+1, \\
\varrho'_j &= 2\pi j + c' + o(1), \quad \varrho''_j = 2\pi j + c'' + o(1), \quad n = 2q,
\end{aligned} \tag{1.9}
$$

where constants c, c', c'' are defined by the leading coefficients $b^0_{\hat{i}}, b^1_{\hat{i}}$ in boundary conditions and $j = \pm N, \pm(N+1), \ldots$. The Green function admits the following remarkable estimate from above off some small δ-neighborhoods of the ch.v. ϱ_j :

$$
G(x, \xi, \varrho) = O(\varrho^{-(n-1)}). \tag{1.10}
$$

Let $S_r(f)$ be the r-th partial sum of e.a.f. expansion including all summands with $|\varrho_j| \le r$:

$$
S_r(f) = S_r(f, L) := -\frac{1}{2\pi i} \int_{\Gamma_r} \int_0^1 G(x, \xi, \varrho) f(\xi) \cdot n\varrho^{n-1} \, d\varrho, \tag{1.11}
$$

$$
\Gamma_r = \{\varrho \in S_0, \quad |\varrho| = r\}. \tag{1.12}
$$

It is suitable to take the integral over contour Γ_r in the principal value sense if it intersects some poles of the Green function, i.e. when some ch.v. $\varrho_j \in \Gamma_r$. In this case the corresponding summands are taken with the factor 1/2.

G.D.Birkhoff has deeply investigated convergence of such expansions for sufficiently smooth functions (f is of bounded variation, $f \in V(0,1)$). It happens so that these expansions behave themselves like the trigonometric ones in all interior points of the main interval $(0,1)$. For instance, the sum (1.11) converges to $\frac{1}{2}(f(x+0) + f(x-0))$. However, at the end points it converges to some linear combinations of the limiting values of f at these points whose coefficients are defined by the boundary forms (1.2).

Further, J.Tamarkin ($p_k \in C(0,1)$) and M.Stone ($p_k \in L(0,1)$) [131, 130, 127] have established the following fundamental result.

Theorem 1.3 (J.Tamarkin-M.Stone). *Let $L \in (R)$ and*

$$
\sigma_r(f) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin r(x-\xi)}{x-\xi} f(\xi) \, d\xi \tag{1.13}
$$

denotes the r-th partial sum of the trigonometric Fourier integral of a given function $f \in L(0,1)$. Then

$$
\lim_{k \to \infty} ||S_{r_k}(f) - \sigma_{r_k}(f)||_{C(K)} = 0 \tag{1.14}
$$

for any compact K in $(0,1)$.

Here the radii r_k are taken in such a way that

$$
dist(\Gamma_{r_k}, \{\varrho_j\}) \ge \varepsilon > 0. \tag{1.15}
$$

For instance, we can take $r_k = 2\pi k + \alpha$ with some appropriate α.

J.Tamarkin also established another useful and important results.

Theorem 1.4 (J.Tamarkin). *Let $L \in (R)$, $f^{(j)}(x)$ be absolutely continuous for $j = 0, \dots, n-1$; $f^{(n)} \in L(0,1)$ and f satisfies boundary conditions (1.2) . Then*

$$\lim_{k\to\infty} \|S_{r_k}(f) - f\|_{C[0,1]} = 0. \tag{1.16}$$

Theorem 1.5 (J.Tamarkin). *Let $L \in (R)$ and denote L^0 an operator defined by the simplest differential expression D^n and the leading boundary conditions (1.2)*

$$V_j(y) = 0, \quad j = 0, \dots n-1. \tag{1.17}$$

Then

$$\lim_{k\to\infty} \left\|S_{r_k}(f, L) - S_{r_k}(f, L^0)\right\|_{C[0,1]} = 0. \tag{1.18}$$

for any function $f \in L(0,1)$.

J.Tamarkin's investigations were summarized in the book [130], published on the eve of the 1917 year revolution in Russia. Therefore its results became known and accessible only after their reprinting in abbreviated and abbriged form in the fundamental article [132].

Of course, there were predecessors for this result, namely such theorem has been earlier established for second order operators in the pioneering works of V.A.Steklov, E.W.Hobson and A.Haar[125, 126, 26, 25].

1.2 Main problems

Theorem 1.3 is remarkable because it completely reduces the question of Birkhoff's series convergence on any internal compact to an analogous one for the model, namely trigonometric system. The latter is a classic problem and it is elaborated during the last two hundred years (if not more). In the meantime theorem 1.3 generated question of the ways of further investigations. In fact, there are four main directions:

1. to consider more general differential expressions, preserving Birkhoff-regularity;
2. to consider more general boundary conditions, including, for instance, those with a Stieltjes integral

$$V_j(y) + \int_0^1 y^{(j)}(x) d\sigma_j(x) = 0, \tag{1.19}$$

   requiring regularity of the leading boundary conditions (1.17), where $d\sigma_j$ denotes a vector-column of height r_j of finite measures which are continuous at the end points $0, 1$;
3. irregular boundary value problems;
4. equiconvergence on the whole interval.

1.3 Polynomial pencils

Observe that the first two problems were investigated at once by J.Tamarkin himself [132]. In particular, he considered a polynomial pencil

$$y^{(n)} + p_1(x,\varrho)y^{(n-1)} + \ldots + p_n(x,\varrho)y = 0, \tag{1.20}$$

$$L_i(y) \equiv \sum_{s=0}^{n} \varrho^s L_i^{(s)}(y) = 0, \quad i = 1, \ldots, n, \tag{1.21}$$

where he put

$$L_i^{(s)}(y) \equiv \sum_{l=1}^{n} \left[a_{il}^{(s)} y^{(l-1)}(0) + b_{il}^{(s)} y^{(l-1)}(1) + \int_0^1 \alpha_{il}(x) y^{(n-1)}(x)\, dx \right] = 0. \tag{1.22}$$

Herewith it is assumed that the so called characteristic equation

$$\varphi^n + p_{10}(x)\varphi^{n-1} + \ldots + p_{n-1,0}(x)\varphi + p_{n0}(x) = 0$$

admits n continuous nonintersecting roots $\varphi_1(x), \ldots, \varphi_n(x)$ together with a lot of other awkward restrictions upon the coefficients in the expression (1.20) and boundary conditions (1.21).

However, observe that under such generality (an integral, containing $y^{(n-1)(x)}$ in (1.22)) many important instances are lost or hidden in contrast with more simple situations when they become transparent. Concretely, the hypothesis that the boundary forms (1.21) are polynomials in ϱ yield a very small (from the first glance) restriction upon the factors α_{il}: *their derivative must be continuous and of bounded variation* [132, p.30]. Then the main part of the characteristic determinant (see (2.2)) of the b.v.p. (1.20)-(1.21) occurs to be a quasipolynomial where the coefficients by the leading exponential terms are nontrivial determinants. Demanding them not to vanish J.Tamarkin just transfers the notion of regularity to this general situation.

Let us, however, assume that these forms don't depend on ϱ, i.e.

$$L_i(y) \equiv L_i^{(0)}(y), \qquad L_i^{(s)}(y) = 0, \quad s > 0. \tag{1.23}$$

Then it is possible to extract the main part of the characteristic determinant only for the normalized boundary conditions (1.17) but not for the general ones (1.21). In other words, in the case (1.23) J.Tamarkin's regularity conditions implicitly demands that these boundary conditions are equivalent to the *standard normalized ones* (1.17) plus, possibly, some lower order terms. Hence such a generality in the case of standard b.v.p. (not pencils) happens to be apparent.

2 Stone-regularity

2.1 Historical remarks

Of course, researchers of the beginning of the century understood very well importance of the problems 1–4 above. However all attempts to investigate irregular two-point

b.v.p. yielded only the class of *decomposing* boundary conditions when m conditions are taken in one end point and $n-m$ in another, $m \neq n-m$, since otherwise such boundary conditions are regular. In the latter case they are called Sturmian conditions and necessarily n is even. It happens so that the Green function of the *decomposing* boundary conditions has an exponential growth in ϱ and the associated e.f. expansions behaves themselves like Taylor series or exponential series in the complex plane. Let us note contribution to the field due to A.P.Khromov, W.Eberhard, G.Freiling, H.Benzinger, B.Schultze and M.Wolter [16, 55, 57, 22, 135]. Earlier articles and extensive bibliography may be found in the book [97] or in the articles just cited.

The problem of finding *good* boundary conditions consists mainly of difficulties with Green's function estimate from below. Namely, the advantage of the resolvent's approach used by G.D.Birkhoff and his successors, leans heavily upon the explicit formula

$$G(x, \xi, \varrho) = \frac{\Delta(x, \xi, \varrho)}{\Delta(\varrho)} \tag{2.1}$$

which stems from the method of variation of constants. At the moment we shall need and recall only a formula for the denominator (which is usually referred to as *the characteristic determinant*):

$$\Delta(\varrho) = |U_j(y_k)|_{j,k=0}^{n-1}\,. \tag{2.2}$$

Here $\{y_j\}_{j=0}^{n-1}$ stands for some f.s.s. of the equation (1.1).

When Birkhoff-regularity conditions are violated we are unable to estimate the characteristic determinant from below for *arbitrary summable coefficients* p_k of the expression (1.1).

In order to get around this difficulty A.P.Khromov in 1962 and H.Benzinger in 1970 introduced a class of S-regular or Stone-regular b.v.p. [54, 8]. Roughly speaking they started to consider not b.v.p. but rather operators because in this approach it is assumed that characteristic determinant admits an asymptotic expansion. Since the only known situation when it is possible is when the coeffients in (1.1) are smooth, $p_k \in C^\infty(0, 1)$, we shall take this hypothesis throughout if otherwise is not explicitly assumed. Of course, only a finite but enough large smoothness is needed but we shall omit here details. Then there exists a f.s.s. with an exponential asymptotics such that exponentials are factored by asymptotic power series in ϱ^{-1}. Then the characteristic determinant happens to be a finite sum of such exponentials. Now it is possible to indicate its main part

$$\Delta(\varrho) = \Delta_0(\varrho) + \ldots, \qquad \varrho \in S_k \tag{2.3}$$

where

$$\Delta_0(\varrho) = \sum c_i(\varrho) \exp(\varrho\sigma_i), \qquad \varrho \in S_k \tag{2.4}$$

and the exponents σ_i have the largest real part. The sum in (2.4) consists of two ($n = 2q + 1$, $i = 1, 2$) or three ($n = 2q, i = 1, 2, 3$) summands. In the even case $\Re\sigma_i > \Re\sigma_{i+1}$. In the odd one σ_1 contains all ε_j such that $\Re(i\varrho\varepsilon_j) \geq 0$ throughout the sector S_k under consideration while σ_2 differs from it by a summand ε_q. The

latter is the unique value such that $\Re(i\varrho\varepsilon_j)$ changes sign in the corresponding ϱ-sector S_k $(0 \le j \le n-1)$.

Call a function $c(\varrho)$ an asymptotic function of order α (α real) in the sector S_k if

$$\exists d = \lim c(\varrho)/\varrho^{\alpha}, \qquad \varrho \to \infty, \qquad \varrho \in S_k \tag{2.5}$$

and

$$d \neq 0. \tag{2.6}$$

Definition 2.1 (A.P.Khromov, H.Benzinger). A b.v.p. is called Stone-regular (shortly S-regular) in the sector S_k if the coefficients $c_i(\varrho)$ in (2.4) are asymptotic power functions of orders α_i, respectively (hence, the limits $d_i \neq 0$). Then the corresponding operator L is called of type (α_1, α_3) if n is even or of type (α_1, α_2) if n is odd.

The Birkhoff's regularity corresponds to the case when

$$\alpha_1 = \alpha_2 = \chi, \quad n \quad \text{odd}; \qquad \alpha_1 = \alpha_3 = \chi, \quad n \quad \text{even}; \qquad ,$$

where the quantity

$$\chi := \sum_{j=0}^{n-1} j r_j \tag{2.7}$$

is called a *total* order of the b.v.p. [120, p.194]. At the first glance this definition depends on the choice of the sector $S_k, k = 1, 2$. However, W.Eberhard and G.Freiling proved independence of the orders α_i of the sector's choice [17]. Later B.Schultze made a complement to the theory adding to the definition of S-regularity the case when $d_2 = 0$ but $\alpha_2 \le (\alpha_1 + \alpha_3)/2$ [111, 112].

Green function of S-regular problem obeys a polynomial estimate

$$G(x, \xi, \varrho) = O(|\lambda|^a). \tag{2.8}$$

Generally speaking, it is *worse* than (1.10). Presently we are able to prove rigorously that $a > -(n-1)/n$ for irregular b.v.p. but this fact falls out of the review's goals and we plan to expose it elsewhere. In view of (2.8) the next theorem looks quite natural.

Theorem 2.2. *[54, Theorem 5] Let L be a S-regular operator of the type (α_1, α_2). If $f \in D_{L^m}$ with $m = \frac{1}{n}$Entier $\left[\sum_{j=0}^{n-1} r_j - \min(\alpha_1, \alpha_2) - (n-1)\right] + 2$, then*

$$\lim_{k \longrightarrow \infty} \|S_{r_k}(f) - f(x)\|_{C(0,1)} = 0 \tag{2.9}$$

The exponent m here is exact.

Advanced results in that theory may be found in in A.A.Shkalikov's articles [120, 121], see also [8]. However, no statements like theorems 1.3,1.5 were proved. In subsection below we give some of A.P.Khromov's results omitting theorems concerning Riesz summability of such expansions.

2.2 Finite functions

At first let us consider the case of equal orders.

Theorem 2.3. *[54, Theorem 5] Let L and L' be S-regular differential operators of types (α, α) and (α', α'), respectively. Given a a number δ, $0 < \delta \leq 1/2$ and a summable function f which vanishes off the interval $K = [\delta, 1-\delta]$ the following relation holds*

$$\lim_{k\longrightarrow\infty} \|S_{r_k}(f, L) - S_{r_k}(f, L')\|_{C(K)} = 0. \tag{2.10}$$

In the second theorem it is assumed that these operators have the same type (α_1, α_2) but these numbers possibly differ.

Theorem 2.4. *[54, Theorem 6] Assume that $c_i(\varrho) = c_i'(\varrho)[1]$, $i = 1, 2$. Then for any summable function f vanishing off some interval $[\delta_1, 1-\delta_2] \subset (0,1)$ the relation (2.10) remains valid. The interval K of equiconvergence is as follows.*

If $|\alpha_2 - \alpha_1| \leq 1$ then $K = [\delta_3, 1-\delta_3]$.

If $|\alpha_2 - \alpha_1| > 1$ then

n	K	ε	$\alpha_2 - \alpha_1$
$\mu + 1$	$[\varepsilon, 1-\delta_3]$	$1 - \|\alpha_2 - \alpha_1\|^{-1} - \delta_2$	$\alpha_2 - \alpha_1 > +1$
$4\mu + 3$	$[\delta_3, \varepsilon]$	$\|\alpha_2 - \alpha_1\|^{-1} + \delta_1$	$\alpha_2 - \alpha_1 < -1$

and it is subject to the evident restrictions that $K \subset (0,1)$ and its left end is less than the right one.

There are also examples demonstrating sharpness of the theorem's conditions. The third theorem deals with a b.v.p. where some of the coefficients p_k vanish:

Theorem 2.5. *[54, Theorem 7] Assume that L is defined by expression*

$$l(y) = y^{(n)} + p_{n-k}y^{(k)} + \cdots + p_n(x)y$$

and similarly for L' with $l'(y)$ of the same type. Suppose also that

$$c_i = c_i' \left[1 + O(\varrho^{k+1-n})\right], \qquad 0 \leq k \leq n - 2 - |\alpha_2 - \alpha_1| \tag{2.11}$$

Implicitly (2.11) implies that both differential operators in question are of the same type (α_1, α_2). Then again (2.10) holds for the function f and the interval K as in the theorem 2.4.

Remark 2.6. The proof of these results is difficult but it is easy to guess the interval of equiconvergence applying our theorem 4.2 (see below).

Of course, it is possible to consider b.v.p. with a polynomial growth of the resolvent from an abstract point of view, i.e. a priori assuming the inequality (2.8) to be fulfilled without regard of how this could be obtained. However, we dare to conjecture that

Conjecture 2.7. *Given a b.v.p. subject to estimate (2.8) with smooth coefficients p_k, then it is Stone-regular. Moreover, we believe that this assertion is also valid for general boundary conditions (1.19).*

2.3 Strongly irregular boundary conditions

Since b.v.p. with decomposing boundary conditions have an exponential growth of the resolvent it is hardly possible to expect any kind of equiconvergence with a trigonometric series expansion. However, B.Schultze succeeded in finding such a phenomenon for a class of b.v.p. which are *partially decomposing*. Further we describe this unexpectable result (see [112]).

Let $0 < m < n, \quad \alpha$ be an $m \times n,\ \beta$ and γ be $(n-m) \times n$ matrices, respectively.

Definition 2.8. [112] A two-point boundary conditions are called *strongly irregular* if they are equivalent to boundary conditions of the form:

$$My^{\vee}(0) + Ny^{\vee}(1) = 0, \qquad y^{\vee}(x) := \left(D^j y(x)\right)_{j=0}^{n-1} \tag{2.12}$$

with

$$M = \left(\frac{\alpha}{\gamma}\right), \qquad N = \left(\frac{0}{\beta}\right), \qquad \text{when} \quad m > n-m \tag{2.13}$$

or with

$$M = \left(\frac{\alpha}{0}\right), \qquad N = \left(\frac{\gamma}{\beta}\right), \qquad \text{when} \quad n-m > m.$$

Here 0 stands for a zero-matrix of appropriate size which is clear from the context.

To be definite we shall confine ourselves in what follows to the first case and consider differential operators L and L' defined by the equation (1.1) and boundary conditions (2.13) (operator L) and

$$M'y^{\vee}(0) + Ny^{\vee}(1) = 0, \qquad M' = \left(\frac{\alpha}{0}\right) \tag{2.14}$$

(operator L').

Definition 2.9. If $A = [a_1, \dots a_m]$ is a $k \times m$-matrix of rank $k, \quad m \geq k$ with columns $a_i (i = 1, \dots m)$, we define the **weight** of the matrix A as in [111, definition 3]:

$$weight(A) := \max \{i_1 + \cdots + i_k \quad | \quad \det [a_{i_1}, \dots, a_{i_k}] \neq 0\}.$$

Theorem 2.10 (B.Schultze). *[112, Remark to Theorem 6]Assume that the weight of the matrix β does not increase, if an arbitrary column of β is replaced by an arbitrary column of the matrix γ. Then the equiconvergence relation (2.10) is valid for any $f \in L(0,1)$ and any compact $K \subset (0,1)$.*

Evidently this theorem has theorem 1.5 as a counterpart and indicates that despite an exponential growth the Green functions of both operators L and L' are very close to one another.

3 General differential expressions

Investigation of the convergence of e.f. expansions at the end points or in their neirborhoods met serious difficulties and has been examined only for sufficiently smooth functions, say for functions of bounded variation, $f \in V[0,1]$ [132, 12].Among the last papers let us note [49]. Therefore writers on the topic concentrated their efforts upon various generalizations of the differential expression $l(y)$ replacing it by

$$l_1(y) = D^n y + Fy \tag{3.1}$$

where F denotes a linear operator dominated in a certain sense by the first summand. Here we want to distinguish the following results.

3.1 Nonsmooth coefficient by the $(n-1)$th derivative

First, let us take

$$l_2(y) = D^n y + Fy, \quad Fy = \sum_{k=0}^{n-1} p_k(x) D^k y. \tag{3.2}$$

If

$$p_{n-1}(x) \in C^{n-1}[0,1], \tag{3.3}$$

then upon substitution

$$y = Vz, \quad V(x) = \exp\left(-\frac{i}{n}\int_0^x p_1(s)ds\right) \tag{3.4}$$

we pass to the standard form (1.1) of the differential expression. However, if (3.3) is broken, namely

$$p_{n-1}(x) \in L[0,1], \tag{3.5}$$

the situation changes abruptly. V.S.Rykhlov established in this case an existence of a f.s.s. with an exponential asymptotics as long as in 1977 [104]. This was a breakthrough in the theory and soon after that he built a nontrivial analogue of the Birhoff's theory. Of course, here also appeared the Birhkhoff-regularity conditions in a slightly modified form: one has only to replace b_j^1 by $b_j^1 \cdot V(1)$ in (1.3)-(1.4). The following main result belongs to this author [105, 106].

Theorem 3.1 (V.S.Rykhlov). *Given a differential operator $L \in (R)$ defined by (3.2) with summable coefficents $p_j(x)$, $j = 0, \ldots, n-1$. Then the equiconvergence (1.14) remains valid provided one of the following relations is fulfilled:*

1. $p_{n-1}(x) \in L^q[0,1], \quad f(x) \in L^p[0,1], \qquad \frac{1}{p} + \frac{1}{q} < 1;$
2. $p_{n-1}(x) \in H_\infty^\alpha[0,1], \quad f(x) \in H_1^\beta[0,1], \qquad \alpha + \beta > 1;$
3. $p_{n-1}(x) \in H_1^\alpha[0,1], \quad f(x) \in H_\infty^1[0,1], \qquad \alpha + \beta > 1.$

Moreover, the following estimate with a modified Dirichlet kernel holds

$$\left\|S_{r_k}(f,L)-\left(V\sigma_{r_k}V^{-1}\right)(f)\right\|_{C(K)} \leq \left(\frac{\log r}{\log^{\alpha+\beta} r}+\frac{1}{\log^{\alpha} r}+\frac{1}{\log^{\beta} r}\right) \tag{3.6}$$

for any compact $K \subset (0,1)$.

Here
$H_1^{\alpha}[0,1] = \left\{g(x) \in L[0,1] \quad | \quad \varpi_1(g,\delta) = O\left(\log^{-\alpha}\frac{1}{\delta}\right),\ \alpha > 0\right\}$,

$H_1^0[0,1] = L[0,1]$;

$H_\infty^{\alpha}[0,1] = \tilde{H}_\infty^{\alpha}[0,1] \qquad (0 \leq \alpha \leq 1)$;

$H_\infty^{\alpha}[0,1] = \tilde{H}_\infty^{\alpha}[0,1] \cup V[0,1] \qquad (\alpha > 1)$;

$\tilde{H}_\infty^{\alpha}[0,1] = \left\{g(x) \in C[0,1] \quad | \ \varpi_1(g,\delta) = O\left(\log^{-\alpha}\frac{1}{\delta}\right),\ \alpha > 0\right\}$

$\tilde{H}_\infty^{0}[0,1] = L_\infty[0,1]$.

Concrete examples clearly demonstrate sharpness of the inequalities $\frac{1}{p}+\frac{1}{q} < 1, \quad \alpha+\beta > 1$ in the theorem's statement. Obviously, (3.6) signifies *an equiconvergence with a rate*. Presently there appeared several new results in this direction, see, for instance [107], but we shouldn't go into further details.

3.2 Integral operators with a Green-type kernel

Next, consider the equiconvergence problem for integral operators of the form

$$Af(x) = \int_0^1 A(x,t)f(t)dt. \tag{3.7}$$

Obviously, any minimal function system in $L^2(0,1)$ occurs to be an e.f. family of some integral operator. Hence, here we deal with a most general situation.

Set

$$A_{s,j}(x,\xi) = \overline{D_\xi^j\left(\overline{D_x^s A(x,\xi)}\right)}, \qquad x \neq \xi \tag{3.8}$$

$$\Delta A_{s,j}(x) := A_{s,j}(x,\xi)\Big|_{\xi=x+0}^{\xi=x-0} \tag{3.9}$$

and suppose the following conditions to be fulfilled:

i) the derivatives $A_{s,j}(x,\xi)$ are continuous whenever $t \leq x$ or $x \leq t$ $(s,j = 0,\dots,n)$;

ii) the jumps $\Delta A_{s,j}(x) \in C^{n-1-j}[0,1]$, $(s,j = 0,\dots,n-1)$;

iii) $A_{s,0}(x,\xi),\ s=0,\dots,n-1$ are continuous and the $(n-1)$th derivative $A_{n-1,0}(x,\xi)$ is discontinuous at the line $x=\xi$

$$\Delta A_{s,0}(x) = i\cdot\delta_{s,n-1}; \quad s=0,\dots,n-1; \tag{3.10}$$

iv) operator A admits no zero e.v.

Then the inverse A^{-1} occurs to be an integro-differential operator of the form

$$l_3(y) = (E+N)\,(D^n y + \alpha y) \tag{3.11}$$

with some boundary conditions

$$U_j(y) = \int_0^1 y(x)\varphi_j(x)dx, \quad j=0,\dots,n-1 \tag{3.12}$$

where E denotes an identity operator in $L^2(0,1)$ and N stands for an integral operator like (3.7) with a kernel $N(x,t)$. The latter is separately continuous in both triangles $t\le x$ or $x\le t$; $U_j(y)$ are n linear independent forms of variables

$$D^j y(0), D^j y(1), \quad j=0,\dots,n-1;$$

α is some complex number. Note that the kernel $N(x,t)$ and the functions $\varphi_j(x)$ may be calculated efficiently through the initial kernel $A(x,t)$.

Hence, any kernel $A(x,t)$ subject to the aforementioned restrictions i)–iv) constitutes a Green function of the integro-differential operator (3.11)–(3.12).

Evidently, the boundary conditions (3.12) may be called *natural*.

Theorem 3.2 (A.P.Khromov). *[58]Assume that*

1) *the integral operator A satisfies restictions i)-iv) and in addition (3.10) is also valid for $s=n$;*

2) *the leading forms $U_j(y)$ in* natural *boundary conditions (3.12) are Birkhoff-regular;*

3) *uniformly in $\xi\in[0,1]$*
$$\underset{0\ \ x}{\overset{1}{Var}}\, A_{n0}(x,\xi) \ \textit{is bounded.} \tag{3.13}$$

Then (1.14) is true where now $S_{r_k}(f)$ stands for the partial sum of the e.f. expansion associated with the operator A. The sum includes all summands whose e.v. are greater than r_k^{-n} in absolute value. Recall that if μ is an e.v. of the integro-differential operator (3.11)–(3.12) then $\lambda=\mu^{-1}$ is an e.v. of A.

If in addition the kernel $A(x,\xi)$ is symmetric, $A(x,\xi)=\overline{A(\xi,x)}$, then condition 2) may be omitted [83]. It merely follows from the regularity of s.a. boundary conditions [109, 21, n even], and [82, n odd].

Conditions of this theorem are exact, no one of conditions 1)–3) may be removed. Add some words concerning apriori restrictions iii)–iv). Condition iv) is needed because otherwise equiconvergence for a function f_0 from the A kernel forces it to be rather specific, namely to have a uniformly convergent trigonometric series. Now about iii). Given an integral operator A with sufficiently smooth kernel, it is possible to indicate an integral operator B retaining the same e.a.f. system and such that iii) is satisfied with some even n.

*Hence, condition iii) singles out a **canonical** operator among all integral operators with the same e.a.f. system admitting equiconvergence with a trigonometric series expansions.*

It is worth noting here that perhaps the first time such an integral operator appeared in R.Langer's article [77] where the case $n = 1$ was treated.

A partial case of integral operators constitute finite convolution operators:

$$Af(x) = \int_0^1 A(x-t)f(t)dt, \quad 0 \le x \le 1. \tag{3.14}$$

From the theorem 3.2 it follows directly

Theorem 3.3 (A.P.Khromov). *[58] Assume that*

1. *the function $A(x) \in C^{2n}$ for $x \ge 0$ and $x \le 0$;*
2. *$D^j A(+0) - D^j A(-0) = i\delta_{j,n-1}, \qquad j = 0, \dots, n;$*
3. *there exists an inverse A^{-1}.*

Then relation (1.14) holds true.

Note that B.V.Pal'tsev has earlier investigated a convolution operator whose kernel coincides with a restriction of the Fourier transform of a rational function [99]:

$$A(x) = \int_{\mathbb{R}} \frac{P(t)}{Q(t)} e^{-ixt} dt, \tag{3.15}$$

where $P(t) = t^p + at^{p-1} + \dots$, $Q(t) = t^q + bt^{q-1} + \dots$ are polynomials in t of orders p and q, respectively.

Theorem 3.4 (B.V.Pal'tsev). *Let $q^+(q^-)$ denote the number of roots of $Q(z)$ in the upper (lower) half-plane. Assume that*

$$n := q - p \ge \min\left\{q^+, q^-\right\}$$

and the polynomial

$$P(t)\overline{Q(t)}, \quad t \in \mathbb{R}$$

has only real coefficents. Define the function $A(x)$ through equality (3.15). Then (1.14) is valid for an integral convolution operator $A_1(f)$ with the kernel $A_1(x-t)$ where

$$A_1(x) = \frac{(-1)^n}{2\pi} A(x) \exp(\theta x), \qquad \theta = \frac{i}{n}(a-b).$$

3.3 Functional–differential perturbation

Let us pass now to a general approach when perturbation F is an abstract linear operator. Let F be a bounded linear operator from the Hölder space $C^\gamma[0,1]$ into the space $L(0,1)$ and impose a restriction

$$0 \le \gamma < n-1. \tag{3.16}$$

Consider a functional–differential operator L defined by (3.1) and boundary conditions (1.19). For a summable function f let $\tilde{f}(x)$ be its extension to the whole axis which vanishes off $[0,1]$. Denote $\varpi^q(\delta, f)$ the qth modulus of continuity of f in $L(0,1)$:

$$\varpi^q(\delta, f) := \sup_{0 \le h \le \delta} \int_{\mathbb{R}} \left| \Delta_h^q \left(\tilde{f}, x \right) \right| dx,$$

$$\Delta_h^q \left(\tilde{f}, x \right) := \sum_{s=0}^{q} (-1)^q C_q^s \tilde{f}(x+hs).$$

G.V.Radzievskii and A.M.Gomilko established

Theorem 3.5 (A.M.Gomilko, G.V.Radzievskii). *[24]Let the boundary forms $V_j(y)$ be Birkhoff-regular. Then*

$$\left\| S_{r_k}(f) - \sigma^\pi_{r_k}(f) \right\|_{C[\delta,1-\delta]} \le d_q \cdot \frac{r_k}{1+r_k\delta} \varpi^q \left(\frac{1}{r_k}, f \right). \tag{3.17}$$

Here $q \equiv 1$ if there are measures in (1.19) and $q = 1, 2, \ldots$ otherwise.

$$\sigma^\pi_r(f) := \sum_{|j| \le r/2\pi} (f, e_j)_{L^2(0,1)} e_j, \quad e_j := \exp(2\pi i j x), \quad 0 \le x \le 1 \tag{3.18}$$

— stands for a partial sum of a trigonometric Fourier series.

Further, G.V.Radzievskii investigated the case of abstract perturbations F acting from $C^\gamma[0,1]$ into $L^p(0,1)$, $\quad 1 \le p < \infty$ or into the Sobolev space W_1^1 [101]. This generalization overlaps such important situation as

$$Fy = Ny^{(n-1)} + \sum_{k=0}^{n-2} p_k(x) D^k y, \tag{3.19}$$

N being an integral operator with a smooth kernel,

$$D^{n-1} N(x,t) \in L_1 \left([0,1] \times [0,1] \right).$$

Herewith he also gave estimates of the rate of convergence of e.a.f. expansions in terms of modulus of continuity of the function in question or in terms of certain special K-functionals.

However, it is necessary to note that V.S.Rykhlov's operator hasn't been covered yet by this approach. It corresponds to the *critical* case

$$\gamma = n-1, \tag{3.20}$$

when the summand Fy can not be viewed of as a *weak* perturbation of the main term $D^n y$. Hence, it is highly desirable to extend their abstract approach to the case (3.20).

3.4 Unconditional equiconvergence

Recently perturbations similar to (3.19) were also investigated by A.G.Baskakov and T.K.Katzaran in [7] from another point of view. Namely, they set

$$Fy = Ky^{(n-1)} + K_0\left(y^{(n-1)} + \alpha y\right), \qquad \alpha \neq 0,\ \alpha \in \mathbb{C}$$

where K denotes a finite-dimensional operator acting from $C(0,1)$ into $L^2(0,1)$ and K_0 is a Hilbert-Schmidt operator in $L^2(0,1)$. They consider a functional-differential operator L defined by (3.1),(3.12) with $\varphi_j \in L^2(0,1)$ and a priori assume that these boundary conditions are strongly regular [97, p.71].

Next, this operator is compared with a model one $\tilde{L}$, generated by the differential expression $y^{(n)}$ with the same boundary conditions. They proved that L is spectral in N.Dunford-J.Schwartz' sense and all its e.v. are simple except, perhaps, for a finite number.

Let $P_\sigma(\tilde{P}_\sigma)$ denote the spectral projector on the portion of the spectrum of operator $L(\tilde{L})$ falling into the set σ. Let us state one of their main results.

Theorem 3.6 (A.G.Baskakov, T.K.Katzaran). *There exists a simultaneous enumeration of the spectra of both operators L and $\tilde{L}$ such that*

$$\left\| P_\sigma - \tilde{P}_{\tilde{\sigma}} \right\|_{L^2(0,1)\to L^2(0,1)} \longrightarrow 0$$

whenever $\min\{|\lambda_j| \quad | \ \lambda_j \in \sigma\} \longrightarrow \infty$.

Note that such phenomenon hasn't been observed earlier even in the case of ordinary differential operators. Let us clarify that spectrality of both operators yields only an estimate from above

$$\left\| P_\sigma \right\|_{L^2(0,1)\to L^2(0,1)}, \left\| \tilde{P}_\sigma \right\|_{L^2(0,1)\to L^2(0,1)} \leq C$$

uniformly with respect to all choices of the subset $\sigma \in \mathbb{R}$, whence

$$\left\| P_\sigma - \tilde{P}_\sigma \right\|_{L^2(0,1)\to L^2(0,1)} \leq 2C.$$

Of course, this is weaker than the theorem's 3.6 assertion.

3.5 General boundary conditions

B.v.p. in a finite interval seem to be useful models of general nonself-adjoint operators. It is especially true for problems with general functionals in boundary conditions. Below we give an account of some known results in this direction related to the equiconvergence problem.

3.6 First order b.v.p.

A.M.Sedletskii [113, 114] investigated a differentiation operator

$$l(y) = y', \qquad -1 \le x \le 1$$

with a difficult to study *smeared* condition:

$$U(y) = \int_{-1}^{1} \frac{k(t)}{(1-|t|)^{\alpha}} y(t)dt = 0 \qquad (0 < \alpha < 1). \tag{3.21}$$

He obtained an equiconvergence on the whole interval with a *shearing* weight.

Theorem 3.7 (A.M.Sedletskii). *If* $\overset{1}{\underset{-1}{Var}} k < \infty, \quad k(1-0)\cdot k(-1+0) \neq 0$, *then for any* $f \in L[-1,1]$

$$\lim_{r \longrightarrow \infty} \|(1-|x|)\cdot [S_r(f) - \sigma_r(f)]\|_{C[-1,1]} = 0. \tag{3.22}$$

Of course, here $r \longrightarrow \infty$ *remaining at least at a fixed positive distance of e.v.* λ_k. *Recall that the latter lies in a strip* $|\Im \lambda_k| \le const$.

For systems of exponentials with a half-bounded spectrum, $\inf \Im \lambda_k > -\infty$, such kind of results as (3.22) was also obtained in [27, Theorem 4.1].

Further S.N.Kabanov [45] studied a differentiation operator with a boundary condition of general form:

$$\alpha y(-1) + \int_{-1}^{1} y'(t)h(t)dt = 0 \tag{3.23}$$

Theorem 3.8 (S.N.Kabanov). *Assume that* $h(t) \in L^q[-1,1]$, $M_1 h(t),\ M_1\tilde{h}(t) \in V[-1,1]$ *where*

$$M_1 h(t) = \int_{-1}^{t} \frac{\partial}{\partial \xi} \frac{(\tau - t)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)} |_{\xi=0}\, h(\tau)d\tau,$$

$\tilde{h}(t) := h(-t)$ *and* $M_1 h(1)\cdot M_1\tilde{h}(1) \neq 0$. *Then for any* $f \in L^p[-1,1], \quad \frac{1}{p} + \frac{1}{q} = 1$, *and for any* $\delta \in (0, 1/2)$ *relation (1.14) is satisfied.*

O.I.Amvrosova [3] considered a higher order operator

$$l(y) = y^{(n)} \tag{3.24}$$

with *smeared* boundary conditions containing power singularities

$$U_j(y) = \int_{-1}^{1} \varphi_j(t)y(t)dt + \int_{-1}^{1} \frac{k_j(t)}{(1-|t|)^{\alpha_j}} y^{(p_j)}(t)dt = 0, \quad j = 1,\dots,n; \tag{3.25}$$

$$n-1 \ge p_1 \ge \ldots \ge p_n \ge 0, \quad 0 < \alpha_j < 1,$$
$$\overset{1}{\underset{-1}{Var}} k_j < \infty, \quad \overset{1}{\underset{-1}{Var}} \varphi_j < \infty,$$

singled out a class of regular boundary conditions and obtained for it the following result.

Theorem 3.9 (O.I.Amvrosova). *Let the boundary conditions (3.25) be regular. Then (3.22) is valid for any function* $f \in L[-1,1]$*. If besides* $f \in L^p[-1,1]$, $p > 1$, $\alpha_\nu > \frac{1}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$ *then*

$$\lim_{r \longrightarrow \infty} \|(1-|x|)^\gamma \left[S_r(f) - \sigma_r(f)\right]\|_{C[-1,1]} = 0 \tag{3.26}$$

for any $\gamma > \frac{1}{p}$.

Later S.N.Kabanov carried over his first order theorem 3.8 to the operator (3.24) [45] with general boundary conditions:

$$U_j(y) = \sum_{k=0}^{n-1} a_{j,k} y^{(k)}(-1) + \int_{-1}^{1} y^{(n)}(t) h_j(t) dt = 0, \quad j = 1, \ldots, n. \tag{3.27}$$

He studied a rather difficult case when the numbers below don't vanish,

$$\beta_j = D_1^{-\alpha_j} h_j(t)_{|t=1} \neq 0, \quad \gamma_j = D_1^{-\alpha_j} \tilde{h}_j(t)_{|t=1} \neq 0, \quad j = 1, \ldots, n, \tag{3.28}$$

where

$$D_1^{-\alpha} h = \frac{1}{\Gamma(\alpha)} \int_t^1 (\tau - t)^{\alpha - 1} h(\tau) d\tau.$$

In this case he singled out a class of regular boundary conditions and arrived at the following result.

Theorem 3.10 (S.N.Kabanov). *Let the leading terms in the boundary conditions (3.27) be regular,* $f \in L^p[-1,1]$, $h_j \in L^q[-1,1]$, $\frac{1}{p} + \frac{1}{q} = 1$*. Then for any* $\delta \in (0, 1/2)$ *relation (1.14) is satisfied.*

For a general first order integro-differential operator

$$(E + N)(y' + \tilde{\alpha} y) \tag{3.29}$$

with general boundary condition (3.23) it is also possible to establish an equiconvergence.

Theorem 3.11 (S.N.Kabanov). *[46]Assume that*

a) $N(x,t) = \sum_1^3 N_i(x,t)$, $N_i(x,t)$ *are continuous for* $t \leq x$ *and* $t \geq x$,

$$N_1(x, x-0) - N_1(x, x+0) = \tilde{\alpha},$$

$$N'_{2,t}(x, x-0) = \varphi(x) s(t), \quad \varphi(x) \in C[-1,1], \quad s(t) \in L^q[-1,1],$$

$$N_3(x,t) = v(x) h(t), \quad v(x) \in C[-1,1];$$

b) for some $\alpha \in (0,1]$ $\quad \underset{-1}{\overset{1}{Var}} D_1^{-\alpha} h(t) < \infty, \quad \underset{-1}{\overset{1}{Var}} D_1^{-\alpha} \tilde{h}(t) < \infty$

c) $D_1^{-\alpha} h(1) \cdot D_1^{-\alpha} \tilde{h}(1) \neq 0.$

Then, if $h \in L^q[-1,1]$, *for any function* $f \in L^p[-1,1]$, $\frac{1}{p}+\frac{1}{q}=1$, *and any* $\delta \in (0,1/2)$ *relation (1.14) is satisfied.*

Necessity of conditions a) here is also justified. More precisely, it is shown that the most general first order integro-differential operator with boundary condition (3.23) can be brought to the form (3.29) where a) is fulfilled.

Next O.I.Amvrosova [4] studied a fractional differentiation operator

$$l(y) = D^\alpha y = \frac{d}{dx}\int_0^x \frac{(x-)^{\alpha-1}}{\Gamma(\alpha)} y(t)dt, \qquad 0<\alpha<1, \quad x\in[-1,1] \tag{3.30}$$

with boundary condition

$$U(y) = \int_{-1}^{1} \frac{k(t)y(t)}{(1-|t|)^{\beta+1}} dt = 0, \tag{3.31}$$

when

$$\overset{1}{\underset{-1}{Var}} k(t) < \infty, \quad 0 < \beta+1 \le \alpha < 1,$$

$$k(0+0) \neq k(0-0), \quad k(-1+0)\cdot k(1-0) \neq 0.$$

Assuming the restrictions above to be fulfilled she obtained

Theorem 3.12 (O.I.Amvrosova). *Let* $f \in L[-1,1]$, $D^\beta f(x)$ *be absolutely continuous in* $[-1,1]$. *Then*

$$\lim_{r\longrightarrow\infty} \left\| (1-|x|)\,|x|^{1+\gamma}\,[S_r(f)(x) - \sigma_r(f)(x)] \right\|_{C[-1,1]} = 0 \tag{3.32}$$

where γ *stands for any positive number.*

At last let us note an interesting A.M.Sedletskii's article [115] where he investigated a uniform convergence of e.f. expansions for a differentiation operator with a Stieltjes integral in boundary condition

$$U(y) = \int_{-1}^{1} y(t)d\sigma(t) = 0,$$

where

$$d\sigma(t) = \frac{b(1-|t|)}{(1-|t|)^\alpha} k(t)dt$$

with a weakly oscillating function $b(t)$. This case is much more difficult then the power singularity case, i.e. when $b(t)\equiv 1$. Hence here opens an opportunity for a movement towards the most general form of boundary conditions involving any kind of singularities at the end points.

Note that this subsection reproduces A.P.Khromov's review on equiconvergence presented at the 7th Saratov winter school in 1994 [60] and is put here under his kind permission.

3.7 Asymptotic formulas for partial sums

In the theorem 1.3 we have no estimates of the rate of equiconvergence. However, in 1967 N.P.Kuptsov [74] has already indicated a possibility of obtaining asymptotic formulas for the remainder term in that theorem. Later he accomplished calculations in the second order case though they have never been published (see remark in [98, p.41]). In this case the asymptotic formulas are very complicated. Therefore in 1973 G.P.Os'kina established such formulas in every subinterval $[\delta, \pi - \delta] \subset (0, \pi)$ in the article [98] accomplished under N.P.Kuptsov's supervisorship. She considered differential expression

$$y^{(n)} + p_{n-2}y^{(n-2)} + \ldots + p_0 y, \quad 0 \leq x \leq \pi$$

and the simplest one, $y^{(n)}$, with one and the same set of boundary conditions (1.2) but at the points 0 and π. Let $S_{r_k}(f)$ and $S^0_{r_k}(f)$ be the corresponding partial sums of e.a.f. expansions and set

$$Qy = l(y) - y^{(n)}.$$

Her main result reads as follows.

Theorem 3.13 (G.P.Os'kina). *Fix $\delta \in (0, 1/2)$. Let n be even and $\delta \leq x \leq \pi - \delta$. Then*

$$S_{r_k}(f) - S^0_{r_k}(f) = \frac{1}{\pi n} \int_0^\pi \int_0^\pi L_{r_k}(x, \xi, t) p_{n-2}(\xi) f(t) dt d\xi + O\left(\frac{1}{r_k}\right) \qquad (3.33)$$

with an explicit though complicated expression for the kernel $L_{r_k}(x, \xi, t)$:

$$L_r(x, \xi, t) = \int_r^\infty \frac{1}{\eta} \left\{\sin \eta \left[|x - \xi| + |\xi - t|\right] + \cos \eta \, |x - \xi| \cdot I_1 - \cos \eta \, |\xi - t| \cdot I_2\right\} d\eta \qquad (3.34)$$

where we set for brevity

$$I_1 = \sum_{j=k+1}^{3k-1} \overline{\varepsilon_j} \exp\left(\eta \varepsilon_j \, |\xi - t|\right), \qquad n = 4k, \qquad (3.35)$$

$$I_2 = \sum_{j=k+1}^{3k-1} \varepsilon_j \exp\left(\eta \varepsilon_j \, |x - \xi|\right), \qquad n = 4k. \qquad (3.36)$$

In the case $n = 4k + 2$ one must take $3k$ as the upper bound in the sums (3.35)-(3.36) and replace there ε_j by $\varepsilon_{j+1/2}$.

4 Equiconvergence and uniform minimality

Eigenfunction systems may also be viewed of as an interesting and important example of families of functions. During the last 20 years such function-theoretic approach has

been elaborated by V.A.Il'in and his school in numerous works, see, for instance, [28]. Below we shall briefly describe some of their results about equiconvergence.

First, consider a function family

$$U = \{u_k\}_{k=1}^{\infty} \tag{4.1}$$

and assume that they are e.f. of the maximal operator generated in $L^2(0,1)$ by the differential expression (3.2) with summable coefficients:

$$lu_k + \lambda_k u_k = \theta_k u_{k-1}, \qquad 0 \le x \le 1, \tag{4.2}$$

where the number θ_k takes two values, 0 — then u_k is an e.f. , or 1—then we require in addition that $\lambda_k = \lambda_{k-1}$ and it is an associated function. Set $\theta_1 = 0$. In the case $n = 1$ we come to an exponential system

$$\{\exp(i\lambda_k x)\}, \quad 0 \le x \le 1 \tag{4.3}$$

— a classical object of the function theory.

4.1 A priori restrictions

To facilitate an exposition let for simplicity n be even and set

$$\mu_k := \left((-1)^{(n+2)/2} \cdot \lambda_k\right)^{1/n} = \left(\varrho e^{i\varphi}\right)^{1/n} = \varrho^{1/n} e^{i\varphi/n}, \quad -\pi < \varphi \le \pi. \tag{4.4}$$

Fix $p \ge 1$ and impose three a priori restrictions:

A1) the system (4.1) is closed and minimal in $L^p(0,1)$,

A2) $|\Im\mu_k| \le C_1$

A3) $\sum_{r \le |\mu_k| \le r+1} 1 \le C_2, \qquad \forall r \ge 0.$

Enumerate all ch.v. μ_k in the ascending modulus order. Observe that A1) yields existence of a unique biorthogonal system

$$\{u'_k\}_{k=1}^{\infty}, \qquad u'_k \in L^q(0,1), \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{4.5}$$

Set

$$S_r(f) = \sum_{|\mu_k| \le r} (f, u'_k) u_k \tag{4.6}$$

Conditions A1)–A3) are assumed to be valid throughout the subsection without further mentioning.

Theorem 4.1 (V.A.Il'in). *In order for the equiconvergence*

$$\lim_{r\longrightarrow\infty}\left\|S_r(f)-\left(V\sigma_r^{\pi}V^{-1}\right)(f)\right\|_{C(K)}=0 \tag{4.7}$$

to be valid for any $f\in L^p(0,1)$ and any compact $K\subset(0,1)$ it is necessary and sufficient that for any compact $K_0\subset(0,1)$

$$\|u_k\|_{L^p(K_0)}\cdot\|u_k'\|_{L^q(0,1)}\le C(K_0). \tag{4.8}$$

All the ingredients in the subtrahend in (4.7) are defined above in (3.4), (3.18).

In particular, equiconvergence holds provided the system (4.1) is uniformly minimal (shortly, $U\in(UM)$), i.e. $K_0=[0,1]$ in (4.8). This result is generalized to the matrix case [15] but we omit the statement due to its awkwardness.

I.S.Lomov [80] investigated equiconvergence on the whole interval for two e.f. subsystems of the form (4.1)–(4.2). To avoid a long introduction let us explain that in the case of two self-adjoint operators L_1,L_2 in $L^2(0,1)$ he arrives at an estimate

$$\|S_r(f,L_1)-S_r(f,L_2)\|_{C[0,1]}\le C\,\|f\|_{V[0,1]} \tag{4.9}$$

for any $f\in V[0,1]$. This estimate stems also from the convergence of both series in question to some limiting values, see discussion in the beginning of the chapter. However, methods developped in the aforementioned paper are powerful and seem to serve well provided boundary conditions would be taken into account.

4.2 General series in eigenfunctions

Given a family (4.1),(4.2) of e.f. , assume that conditions A2)–A3) are fulfilled and omit the first one: A1). Consider a general series

$$\sum_{k=1}^{\infty}c_ku_k(x) \tag{4.10}$$

and assume that it converges to some summable function f on subinterval $J\subset(0,1)$ in the weak sense:

$$\lim_{r\to\infty}\int_J S_r(x)\overline{\varphi(x)}dx=\int_J f(x)\overline{\varphi(x)}dx \tag{4.11}$$

for any $\varphi(x)$ such that $\varphi,l^*(\varphi)\in L^2(J)$. Here

$$S_r(x):=\sum_{|\mu_k|\le r}c_ku_k(x).$$

Assume also that

$$c_k\cdot\|u_k\|_{L^2(J)}\longrightarrow 0\quad as\quad \mathrm{k}\to\infty. \tag{4.12}$$

Theorem 4.2 (A.M.Minkin). *[86]Under the conditions (4.11)-(4.12) the following relation holds*

$$\lim_{r \longrightarrow \infty} \|S_r(x) - \sigma_r(f)\|_{C(K)} = 0 \tag{4.13}$$

for any compact $K \subset J$. Here f is extended as 0 to the whole axis off J.

Compare two preceding theorems in the case $p_{n-1}(x) \equiv 0$. In the theorem 4.2 completeness is omitted. Convergence in some weak sense is used instead. Minimality isn't required at all. An analogue of uniform minimality — inequality (4.8)— is replaced by an obviously necessary and very weak condition (4.12). The advantages are evident: the domain of equiconvergence of a given e.f. expansion is easily to determine through the condition (4.12). Of course, (4.8) with $K_0 = J$ yields (4.12) but not the converse.

Note also that E.I.Moiseev [95, 96] has accomplished a deep investigation of concrete sine/cosine and exponential systems in $L^p(0, \pi), 1 < p < \infty$:

1. the system (4.3) with $\lambda_k = k - \frac{\beta}{2} sign\ k, \quad k \in \mathbb{Z}$;
2. general systems $G_{\beta,\gamma} := \{\sin((k + \beta/2)x + \gamma/2)\}_{k=1}^{\infty}$
3. sine systems $S_\beta := \{\sin((k + \beta/2)x)\}_{k=1}^{\infty}$
4. cosine systems $C_\beta = 1 \cup \{\cos((k + \beta/2)x)\}_{k=1}^{\infty}$

He established delicate estimates of their biorthogonal systems using difficult calculations. From his results it follows that $S_\beta \in (UM)$ if $> \frac{1}{p} - 2$ but is complete only if $\beta \le \frac{1}{p}$.

Analogously, $C_\beta \in (UM)$ if $> \frac{1}{p} - 1$ but is complete only if $\beta \le \frac{1}{p} + 1$. We omit formulation for the general system $G_{\beta,\gamma}$ to shorten an exposition, see details in [96].

Note also the paper [41] of V.A.Il'in and E.I.Moiseev where an important partial case ($\beta = 0$) of the system $G_{\beta,\gamma}$ was considered where the corresponding function family happens to be a mixture of two sets of e.f. of distinct b.v.p. .

Hence, V.A.Il'in's theorem 4.1 holds for any $f \in L^p(0, \pi)$ if the parameter β is such that the corresponding system is complete and uniformly minimal.

However, for $p = 2$ even a stronger result is valid. Namely, equiconvergence with a trigonometric series expansion holds for any f in the span of the corresponding system in $L^p(0, \pi)$ if the parameter β is such that the corresponding system is only uniformly minimal, see our theorem 4.4 below.

Note that actually theorem 4.4 is true for any $p, \ 1 < p < \infty$ but this generalization needs *almost orthogonality* of the Birkhoff's f.s.s. in $L^p(0, 1)$ which is valid but we haven't yet published this result.

Theorem 4.1 poses a natural question:

when and under what assumptions do conditions A1)–A3) hold?

This problem seems to be very difficult. Observe that these requirements are obviously fulfilled for strongly regular two-point b.v.p. as well as for unconditional bases from exponentials (4.3) with the A3) condition being fulfilled. Complete description of the latter systems with **arbitrary** exponents λ_k is given in [88].

For the beginning it would be important to write down explicitly classes of irregular two-point b.v.p. satisfying A1)–A3), at least separately.

A deep problem of uniform minimality of e.a.f. families also deserves a separate treatment. It seems that E.I.Moiseev's results cited above give a certain foundation for our conjecture about connection between basicity and uniform minimality.

At first, let us introduce a distance between two normalized in $L^p(0,1),\ 1<p<\infty$, e.f. families $U := \{u_k\},\ \tilde{U} := \{\tilde{u}_k\}$, satisfying (4.2):

$$d(U,\tilde{U}) := \sup_k \left[\sum_{j=0}^{n-1} \left(|u_k^{[j]}(0) - \tilde{u}_k^{[j]}(0)| \right) + |\mu_k - \tilde{\mu}_k| \right].$$

Conjecture 4.3. *Call the system U **stably** complete (incomplete) if there exists a small $\varepsilon > 0$ such that completeness (incompleteness) preserves for any other e.f. system $\tilde{U}$ with distance $\le \varepsilon$. Then*

- *An e.f. family (4.1) is a basis in $L^p(0,1)$ if and only if it is uniformly minimal and stably complete,*
- *it is a basis in its span if and only if it is uniformly minimal and stably incomplete,*
- *in the case $p=2$ basicity should be replaced by unconditional basicity.*

Let us also indicate how to get around the condition (4.11).

Theorem 4.4 (A.M.Minkin). *Assume that the family U of e.a.f. (4.1),(4.2) is uniformly minimal in $L^2(0,1)$, let $U' = \{u'_k\}$ be some its biorthogonal system which obeys an inequality*

$$\|u_k\|_{L^2(0,1)} \cdot \|u'_k\|_{L^2(0,1)} \le C, \quad \forall k.$$

It is not important, whether U is complete or not. Denote $E = spanU \subset L^2(0,1)$, let

$$S_r(f) := \sum_{|\mu_k|\le r} (f, u'_k) u_k$$

be a partial sum of the corresponding e.f. expansion. Then (4.7) remains true for any $f \in E$ and for any compact $K \subset (0,1)$.

Proof. Fix a compact $K \subset (0,1)$ and at first assume that f is a linear combination of e.a.f. . Then considerations in the theorem 4.2 yield an upper estimate:

$$|S_r(f)(x) - \left(V\sigma_r^\pi V^{-1}\right)(f)(x)| \le C_1 \cdot \sup_k |(f,u'_k)| \cdot \|u_k\|_{L^2(0,1)}, \quad x \in K.$$

Here we need no assumption (4.11) because the series in question reduces to a finite sum. Next, using uniform minimality we immediately derive that this difference constitutes a uniformly bounded family of linear operators acting from $L^2(0,1)$ to $C(K)$. But on a dense (in E) subset of linear combinations of e.f.

$$\left(V\sigma_r^\pi V^{-1}\right)(f) \longrightarrow f$$

in $C(K)$ because e.a.f. are enough smooth. It remains to apply the Banach-Steinhaus theorem. ∎

5 Singular self-adjoint operators

5.1 Self-adjoint expressions

First recall that given a differential expression (3.2) in some interval $G = (a, b)$ of the real axis its formally adjoint or Lagrange-adjoint is defined as follows

$$l^*(z) = z^{\{n\}} = Dz^{\{n-1\}} + \overline{p_0(x)} z^{\{0\}}, \tag{5.1}$$

where

$$z^{\{0\}} = z, \quad z^{\{j\}} = Dz^{\{j-1\}} + \overline{p_j(x)} z^{\{0\}}, \qquad j = 1, \ldots, n-1. \tag{5.2}$$

It remains a *differential* expression provided coefficients are enough smooth

$$p_j^{(j)} \in L^1_{loc}(G). \tag{5.3}$$

Hence in the theory of s.a. differential operators it is natural to write the operation $l(y)$ in the form ($q = Entier(n/2)$)

$$l(y) = D^n y + \sum_{\mu=0}^{q-1} \left(l_{2\mu}(y) + l_{2\mu+1}(y) \right); \tag{5.4}$$

$$l_{2\mu}(y) = D^\mu \left(f_\mu(x) D^\mu y \right), \qquad l_{2\mu+1}(y) = iD^\mu \left\{ Dg_{n-\mu} + \overline{g_{n-\mu}} D \right\} D^\mu y.$$

The coefficients $f_\mu(x), g_{n-\mu}$ are locally summable. Therefore (5.4) is understood as a quasidifferential expression (q.d.) (see, [97, Ch.V]). The most general form of s.a. expressions is given below in chapter 3 (formula (3.0.1) there). Further we assume that $l(y)$ is s.a. , $l^* = l$. Denote $L_{\min}$ the minimal symmetric operator in $L^2(G)$ obtained by restricting $l(y)$ to the set of smooth enough functions with compact support and then taking its closure.

5.2 Classification

Recall that in the theory of s.a. operators a differential expression $l(y)$ is called *regular* (don't mix with the Birkhoff-regularity) if the following two conditions are fulfilled:

i) the interval G is finite;

ii) all the coefficients of $l(y)$ are summable on G.

Otherwise it is called *singular*.

In the regular case one needs to add n s.a. boundary conditions at the end points in order to define a s.a. differential operator in $L^2(G)$. However, it is now known that such b.v.p. are Birkhoff-regular (see [109, 21] if n even and [82] if n is odd). A short proof which is valid for all n at once is given in [89]. Hence, in the s.a. case equiconvergence obviously holds. Now let us pass to singular expressions. First of all recall some basic facts from the abstract spectral theory.

5.3 Spectral function

In the singular case the number n_λ of solutions from $L^2(G)$ of the equation

$$l(y) = \lambda y \tag{5.5}$$

is called a defect number. It is stable in the upper/lower half-plane $\mathbb{C}_\pm,\ n_\lambda \equiv n_\pm,\ \lambda \in \mathbb{C}_\pm$. The pair (n_+, n_-) is called *a defect index* of the differential expression $l(y)$.

If they coincide, $n_+ = n_- = m$, then there exists a s.a. extension L in $L^2(G)$ of the symmetric operator $L_{\min}$ defined by m s.a. boundary conditions. However, the latter should be understood in some special sense (see details in [97, Ch.V]).

Let $E_t, -\infty < t < \infty,\ E_t \le E_s,\ t < s$ be the corresponding resolution of identity. Normalize it as folows

$$E_t = \frac{1}{2}\left(E_{t+0} + E_{t-0}\right), \qquad \lim_{t\to-\infty} E_t = 0, \quad \lim_{t\to\infty} E_t = I.$$

In the sequel these conditions are assumed to be fulfilled without further mentioning.

The ortoprojector E_t turns out to be an integral operator with a kernel

$$\Theta(x,s,t), \quad x,s \in G, \qquad -\infty < t < \infty$$

see [13, Chapter 13]. There exists an explicit formula for the spectral function Θ but it shouldn't be needed in theorems' statements and therefore omitted. If

$$n_+ \ne n_- \tag{5.6}$$

then there also exists a s.a. extension L^+ of $L_{\min}$ but in some ambient space H^+, containing $H = L^2(G)$ as a proper subspace. Let E_t^+ be the resolution of identity associated with L^+ and P^+ be the ortoprojector onto H in H^+. Setting

$$F_t := P^+E_t^+|H$$

we obtain a nonorthogonal resolution of identity. F_t is also an integral operator and its kernel $\Theta(x,s,t)$ is called a generalized spectral function (g.sp.f.), the same title is also applied to the family F_t itself but usually this fact causes no ambiguity. The only difference between E_t and F_t is that instead of the Parceval equality F_t obeys the Bessel inequality

$$\int_{\mathbb{R}} d_t(F_t f, f) \le (f,f), \qquad \forall f \in H.$$

Inequality (5.6) may realize not only for odd n but also in the even case provided the coefficents of $l(y)$ are complex-valued functions.

5.4 Schrödinger operator

To begin with let $G = \mathbb{R}$, $\quad n = 2$, $\quad q(x)$ be a real valued measurable function, $q \in L^1_{loc}(G)$ and let

$$l_4(y) = -y'' + q(x)y.$$

Let $\Theta_0(x,s,t)$ be a sp.f. , corresponding to the zero potential, $q(x) \equiv 0, \quad G = \mathbb{R}$, namely

$$\Theta_0(x,s,t) = \begin{cases} \frac{1}{\pi}\frac{\sin r(x-s)}{x-s}, & r=\sqrt{t}, \quad t>0, \\ 0 & t \le 0. \end{cases} \tag{5.7}$$

Theorem 5.1 (B.M.Levitan). *Fix a compact $K \subset \mathbb{R}$. Then*

$$\lim_{t \longrightarrow +\infty} [\Theta(x,s,t) - \Theta_0(x,s,t)] = 0 \tag{5.8}$$

uniformly with respect to $x, s \in K$.

Next, consider a half-bounded interval $G = [0, \infty)$ and assume that a real valued function $q \in L[0,a]$ for any $a > 0$. Since the potential is real the defect index equals $(1,1)$ or $(2,2)$. In the first case add to equation

$$l_4(y) = \lambda y \tag{5.9}$$

a boundary condition at the origin

$$y'(0) = hy(0), \quad h \neq \infty, \quad h \text{ real.} \tag{5.10}$$

In the second case add in addition a boundary condition at infinity (we omit details).

Denote $\Theta^h(x,s,t)$ the sp.f. of the problem (5.9)–(5.10) and let $\Theta_1^h(x,s,t)$ be a sp.f. of the same problem with a zero potential.

Theorem 5.2 (B.M.Levitan, V.A.Marchenko). *Let $b > 0$. Then uniformly with respect to $x, s \in [0,b]$*

$$\lim_{t \longrightarrow +\infty} \left[\Theta^h(x,s,t) - \Theta_1^h(x,s,t)\right] = 0, \tag{5.11}$$

$$\lim_{t \longrightarrow +\infty} \left[\Theta_1^h(x,s,t) - \Theta_0(x,s,t)\right] = I_1 + I_2 + I_3, \tag{5.12}$$

where we set

$$I_1 = \frac{h}{\pi}\int_0^\infty \frac{\sin\nu(x+s)}{\nu^2+h^2}\nu \, d\nu,$$

$$I_2 = -\frac{2h^2}{\pi}\int_0^\infty \frac{\cos\nu(x-s)}{\nu^2+h^2} \, d\nu,$$

$$I_3 = \begin{cases} 0, & h \ge 0, \\ h^2 e^{h(x+s)}, & h < 0. \end{cases}$$

When $h = 0$ the boundary condition (5.10) should be understood as

$$y(0) = 0. \tag{5.13}$$

Together with $q(x) \equiv 0$ it corresponds to the sp.f.

$$\Theta_1^\infty(x,s,t) = \frac{2}{\pi}\int_0^r \sin\nu x \cdot \sin\nu s \, d\nu, \qquad t > 0 \tag{5.14}$$

and $\Theta_1^\infty \equiv 0, \quad t \le 0$.

Theorem 5.3 (B.M.Levitan, V.A.Marchenko). *Let $b > 0$. Then uniformly with respect to $x, s \in [0, b]$*

$$\lim_{t \longrightarrow +\infty} [\Theta^{\infty}(x, s, t) - \Theta_1^{\infty}(x, s, t)] = 0 \tag{5.15}$$

Conditions of the theorems 5.1–5.3 means exactly equiconvergence for δ_s— the **delta-function** at the point s. Moreover, uniformity of their statements with respect to $s \in [0, b]$ means that equiconvergence also holds for measures with finite support

$$\lim_{r \longrightarrow \infty} \left\| S_r(d\mu) - S_r^0(d\mu) \right\|_{C[0,b]} = 0, \qquad \forall b > 0. \tag{5.16}$$

Here $S_r^0 = \int_{-r^n}^{r^n} dE_t^0$ where the resolution of identity E_t^0 corresponds to the case of zero potential. In particular, it is possible to take $f \in L[0, b], \quad f \equiv 0, \quad x > b$ in (5.16) instead of $d\mu$.

Note that the same assertion is valid for $f \in L^2(G)$. Concretely,

Theorem 5.4 (B.M.Levitan, V.A.Marchenko). *Let either $G = \mathbb{R}$ or $G = [0, \infty)$ and in the latter case the boundary conditions (5.10) or (5.13) are added to the equation (5.9). Then for any compact $K \subset G$, the regular end point 0 included (if present)*

$$\lim_{r \longrightarrow \infty} \left\| S_r(f) - S_r^0(f) \right\|_{C(K)} = 0. \tag{5.17}$$

Remark 5.5. Observe that in the theorems 5.2–5.4 the difference of sp.f. or e.f. expansions vanishes uniformly up to one (regular) end point. This is much more subtle fact than the question of equiconvergence on internal compacts which may also be established by the methods of the articles [78, 81] as well.

As far as we learned from our elder colleagues an equiconvergence theorem for singular s.a. operators was also obtained by N.P.Kuptsov independently of B.M.Levitan and V.A.Marchenko. However, his proof has never been published and seems to be irrevocably lost.

5.5 Higher order

Constructions of B.M.Levitan and V.A.Marchenko extensively used the theory of hyperbolic equations as well as the theory of the transformation operators. Therefore it was impossible to transform them directly to the high-order case. Only ten years later A.G.Kostuchenko made a breakthrough and succeeded to generalize their results to the even order s.a. operators [70, 69]. Unlike [78, 81] he bypassed a use of the transformation operators theory which are *unbounded* for $n > 2$. Instead, he applied the theory of parabolic equations

$$\frac{\partial u}{\partial t} = l(u),$$

where l is a s.a. differential expression of the form (1.1).

Theorem 5.6 (A.G.Kostuchenko). *Let $G = \mathbb{R}$ and $L_{\min}$ be defined by a s.a. expression (1.1) of order $n = 2q,\ q > 1$ with real coefficients. In addition, assume that $L_{\min}$ is half-bounded and*

$$p_{n-2}(x) \text{ is a piece-wise smooth function, } \ p_j \in L^1_{loc}(G), \quad 0 \le j < n-2. \tag{5.18}$$

Let L be its s.a. extension in $L^2(G)$ with a sp.f. $\Theta(x, s, t)$. Set $r := t^{1/n},\ t > 0$. Then the theorem's 5.1 assertion is valid.

Theorem 5.7 (A.G.Kostuchenko). *Let $G = [0, \infty)$ and all other theorem's 5.6 assumptions be satisfied. Assume also that 0 is a regular end point, i.e. $p_j \in L[0, a)$, $\forall a > 0$ and the defect index equals (q, q). Then any s.a. extension L of $L_{\min}$ is generated by q s.a. boundary conditions at 0 [97, pp.212–214]:*

$$B_j(y) \equiv \sum_{i=1}^{n} b_{ij} y^{(j-1)}(0) = 0, \qquad j = 1, \ldots, q; \tag{5.19}$$

$$\sum_{j=1}^{q} b_{ij}\overline{b_{k,n+1-j}} - \sum_{j=1}^{q} b_{i,n+1-j}\overline{b_{kj}} = 0, \quad i, k = 1, \ldots q. \tag{5.20}$$

A.G.Kostuchenko also assumes these forms to obey two additonal complicated restrictions which we shall omit for simplicity. Denote $\Theta_1(x, s, t)$ a sp.f. of a model s.a. operator generated by the expression $D^n y$ and s.a. boundary conditions (5.19). Then uniformly with respect to $x, s \in [0, b]$

$$\lim_{t \longrightarrow +\infty} [\Theta(x, s, t) - \Theta_1(x, s, t)] = 0 \tag{5.21}$$

for any $b > 0$.

He also established equiconvergence for square summable functions. The statement repeats that of the theorem 5.4 and therefore is omitted here.

It is difficult to underestimate the importance of his contribution. However, observe that self-adjointess of $l(y)$ yields some smoothness of the coefficients in addition to (5.18) as is readily seen from (5.1)-(5.4). Moreover, A.G.Kostuchenko treats the general case of nonhalf-bounded $L_{\min}$ by its squaring. This operation requires existence of additional n derivatives of each of the coefficients p_j.

At the beginning of eighties we obtained equiconvergence theorems for singular s.a. high-order equations *without any unnecessary a priori restrictions.*

Theorem 5.8 (A.M.Minkin). *[92]Let G be a finite or infinite interval of $\mathbb{R}$, $l(y) = y^{[n]}$ be a general quasi-differential s.a. expression of the form*

$$y^{[0]} = y, \quad y^{[1]} = Dy^{[0]}, \tag{5.22}$$

$$y^{[j]} = Dy^{[j-1]} + \sum_{k=0}^{j-2} p_{j-1,k}(x) y^{[k]}, \quad j = 2, \ldots, n$$

with complex-valued coefficients such that

$$p_{i,k} \in L^1_{loc}(G), \qquad p_{i,k}(x) = \overline{p_{n-1-k,n-1-i}(x)}. \tag{5.23}$$

Let $L_{\min}$ be the corresponding minimal symmetric operator in $L^2(G)$ and F_t be some its g.sp.f. . For instance, F_t may be a restriction of a sp.f. of some s.a. extension of $L_{\min}$ in a larger interval $G_1 \supset G$. Set

$$\overset{0}{L^1}(G) := \{f \in L^1(G) \quad | \quad f \equiv 0 \quad \textit{near the boundary}\}$$

Let $f \in \overset{0}{L^1}(G) \cup L^2(G), \quad g \in L^2(G) \cup L^1(\mathbb{R})$ and assume that $f(x) \equiv g(x)$ for almost all $x \in \Omega = (a,b) \subset G$. Then

$$\lim_{r\to\infty} \left\| \int_{-r^n}^{r^n} dF_t f - \sigma_r(g) \right\|_{C(K)} = 0 \tag{5.24}$$

for any compact $K \subset \Omega$.

Notice that (5.24) means simultaneously equiconvergence and localization. The case $f \in L^2(G)$ appeared for the first time in [85]. Sketch of the proof is published in [84] provided that the operator measure dF_t is discrete. Proof of the general case closely follows the lines of the discrete one, see [92].

For $n = 2q$ and $G = [0,\infty)$ this theorem's statement may be improved in order to include the end point 0. Namely, require that

$$p_{i,k} \in L[0,a), \qquad \forall a > 0$$

and assume that the s.a. expression $l(y) = y^{[n]}$ has a defect index (m,m) in $L^2(G), \quad q \leq m \leq n$. Since the coefficients are complex this is really a requirement (see, [1, p.175]). A deep investigation of defect indices for general symmetric systems of singular differential equations has been accomplished in an article of V.I.Kogan and F.S.Rofe-Beketov [62]. We refer the interested reader to it for more information and details.

Theorem 5.9 (A.M.Minkin). *Let L be a s.a. operator in $L^2[0,\infty)$, defined by (5.22) and m boundary conditions at least q from which are given at the origin like (5.19). Fix $b > 0$ and let $f \in L[0,b], \quad f(x) \equiv 0, \ x > b$. Then for any $\varepsilon \in (0,b)$*

$$\lim_{r\to\infty} \|S_r(f,L) - S_r(f,L_b)\|_{C[0,b-\varepsilon]} = 0 \tag{5.25}$$

where L_b stands for an ordinary s.a. differential operator in $L^2[0,b]$ generated by $l(y)$ and decomposing s.a. boundary conditions q of which coincide with (5.19) and the other q are taken at the end point b.

Let us stress the fact that in the theorem 5.8 the coefficients' requirements are the least possible (see, (5.23)). Moreover, it covers at once q.d. equations of arbitrary order as well as s.a. extensions going outside the space $L^2(G)$. Earlier these questions haven't been considered at all. In addition theorem 5.9 removes unnecessary restrictions on the coefficients of the boundary forms $B_j(y)$ imposed in [69].

Later V.I.Imamberdiev established a sp.f. asymptotics of an odd order s.a. operator developing the parabolic equations method, see [42].

5.6 Kato condition

Looking closely at the theorems 5.2–5.4 as well as at theorems 5.7, 5.9 one sees that they doesn't constitute a *full* generalization of Tamarkin's theorem 1.5. Indeed, only *one end point* is included! Therefore it is quite natural to pose a question of whether these results are valid throughout the *whole infinite interval* G.

Even in the second-order case this problem remains open in the case of arbitrary potential $q(x)$. However, recently this important question has been answered in affirmative in a series of articles due to V.A.Il'in, I.Antoniu and L.V.Kritskov [36]–[39].

They considered a class of potentials in $\mathbb{R}$ satisfying **Kato** condition:

$$\sup_{-\infty<x<\infty} \int_x^{x+1} |q(s)|ds \leq C. \tag{5.26}$$

Let us state their results.

Theorem 5.10 (V.A.Il'in). *[35]Consider a s.a. operator L in $L^2(\mathbb{R})$ with potential $q(x)$ satisfying Kato condition. Let $\Theta(x,s,t)$ be its sp.f. and define $\Theta_0(x,s,t)$ as in (5.7). Then there exists $T>0$ such that for some finite constant C_T*

$$\sup_{t\geq T}\sup_{x,s\in\mathbb{R}} |\Theta(x,s,t)-\Theta_0(x,s,t)| = C_T. \tag{5.27}$$

In addition if $1\leq p\leq 2, \quad f\in L^p(\mathbb{R})$ then

$$\lim_{r\longrightarrow\infty} \|S_r(f)-\sigma_r(f)\|_{C(\mathbb{R})} = 0. \tag{5.28}$$

This theorem was preceded by result due to I.Antoniu and V.A.Il'in [36] where the Hill operator ($q(x)$ is continuous periodic function on $\mathbb{R}$) was investigated. Afterwards the theorem 5.10 was carried over to the Schrödinger operator with a matrix potential $Q(x)$ satisfying Kato condition [76]. A.V.Kurkina proved an inequality

$$\begin{aligned} \sup_{t\geq T}\sup_{x,s\in\mathbb{R}} \sum_{k=1,k\neq j}^{m} \{|\Theta_{jk}(x,s,t)| + |\Theta_{jj}(x,s,t)-\Theta_0(x,s,t)|\} \quad &\leq \quad C_T<\infty, \\ j \quad &= \quad 1,\dots,m \end{aligned}$$

for the components $\Theta_{jk}(x,s,t)$ of the sp.f. $\Theta(x,s,t)$.

Further V.A.Il'in and I.Antoniu investigated the so called liouvillian, generated by a s.a. Schrödinger operator satisfying Kato condition [37]. It is important for physical applications but we have to omit the statement because it requires introducing a lot of preliminary notions.

Of course, it would be important to generalize these results to higher orders as well as to clarify necessity of the Kato condition in the question of equiconvergence on the whole interval G.

6 Multidimensional Schrödinger-type operator

During the past 20 years the author developped a rather general approach to equiconvergence problems. It will be discussed in other chapters. Here we bring to the reader's attention its application to operators in partial derivatives obtained in a joint article with L.A.Shuster [93].

Let $D = [-1, 1]^m,\ m > 1,\ q(x)$ be a real valued summable function in D. Set

$$Ly = (-\Delta)^n y + q(x)y, \qquad \Delta = \frac{\partial}{\partial x_1^2} + \ldots + \frac{\partial}{\partial x_m^2}.$$

At first L is defined on trigonometric polynomials

$$e_s(x) := \exp(i\pi \langle s, x\rangle), \quad \langle s, x\rangle = \sum_{j=1}^{m} s_j x_j, \quad s \in \mathbb{Z}^m.$$

Under restriction $2n > m$ there exists its Friedrichs extension which will also be written as L and happens to be a half-bounded s.a. operator in $L^2(D)$, satisfying periodic boundary conditions $y(x+2s) = y(x),\ \ s \in \mathbb{Z}^m$. Denote its spectrum $\sigma(L)$ and let $\varphi(k)$ be the number of integer solutions from $\mathbb{Z}^m$ of the equation

$$|s|^2 = k,\ k \text{ natural}, \tag{6.1}$$

where $|s|^2 := \langle s, s\rangle$. We shall need the following

Definition 6.1. For any continuous 2-periodic function on D set

$$\hat{f}(s) := \int_D f(x) \cdot \overline{e_s(x)} dx,$$

$$\|f\|_A := \sum_{s\in\mathbb{Z}^m} \left|\hat{f}(s)\right|.$$

Introduce a space A of all absolutely convergent series on $\mathbb{R}^m/(2\mathbb{Z})^m$ as the subspace of continuous 2-periodic function f on D such that the A-norm is finite, see [47].

It is known that in absence of the potential $q(x)$ the Laplace operator $-\Delta$ on the torus $\mathbb{R}^m/(2\mathbb{Z})^m$ has e.v. of high multiplicity. In the presence of potential, generally speaking, such a multiple e.v. splits into a group of neighboring e.v. . The theorem below is due to L.A.Shuster and gives a rigorous description of this phenomenon.

Theorem 6.2 (L.A.Shuster). *Take $a \in (0, \frac{1}{2}\pi^2)$ and let $2n > m + 3$. Then*

1. *there exists an integer $k(a)$ such that*

$$Card\left\{\lambda \in \sigma(L) \mid \left|\lambda^{1/n} - k\pi^2\right| \le a\right\} = \varphi(k), \forall k \ge k(a). \tag{6.2}$$

2. *Denote $H(k)$ the spectral subspace of L spanned by all e.f. with e.v. in the cluster (6.2). If in addition $n > m+1$ than there exists a basis $\{h_j(x)\}_{j=1}^{\varphi(k)}$ in $H(k)$ such that*

$$h_j(x) = \exp(i\pi \langle s, x\rangle) + O\left(k^{-(n-m-1)}\right), \tag{6.3}$$

where $s \in \mathbb{Z}^m$ runs over all solutions of the equation (6.1). The symbol $O()$ is understood here in the A-norm sense and the constants are absolute.

Under this theorem's assumptions let P_k be the ortoprojector onto $H(k)$ in $L^2(D)$ and set

$$\tau_r(f) := \sum_{k(a)\le k \le r^2} P_k f.$$

Denote P_0 an ortoprojector onto the set of all e.f. of L corresponding to a finite number of first e.v. λ such that $\lambda < \left(k(a)\pi^2 - a\right)^n$ and set $S_r(f) := P_0(f) + \tau_r(f)$. In addition denote

$$\sigma_r^\pi(f) = \sum_{|s|\le r} (f, e_s)e_s$$

the r-th partial sum of a multiple trigonometric Fourier series.

Theorem 6.3 (A.M.Minkin). *Assume that $2n > m+3$. Then $\forall f \in L^2(D)$*

$$\lim_{r\to\infty} r^{2n-2m-3/2} \left\|S_r(f) - \sigma_r^\pi(f)\right\|_A = 0. \tag{6.4}$$

Obviously, the theorem's assertion claims equiconvergence with rate provided that $2n > m+3$ and divergence with rate, otherwise. Moreover, it is established in the A-norm which is principally stronger than the C-one. Note also that equiconvergence holds for *nonsmooth, namely, square summable* functions. Recall that known results requires considerable order of the Riesz typical means of the function in question, (see [2, p.70–76]).

It seems to us natural to join together e.f. corresponding to the same cluster. However, it is likely that this idea hasn't been employed earlier in the spectral theory of operators in partial derivatives. We think that strong smoothness requirements on the function f usual in that theory stem from the fact that one tries to obtain convergence of the series itself and not of an appropriate series *with brackets*, instead.

7 General equiconvergence principles

In this section we briefly outline several general approaches for the equiconvergence problem.

7.1 Iteration of the resolvent's equation

Given a Banach space B with a norm $\| \ \|_B$, a dense lineal $D \subset B$ endowed with a second norm $\| \ \|_D$, consider linear operators A and Q mapping D into B. Note that D

isn't necessarily closed in the norm $\| \ \|_D$. Consider a family of concentric circonferences C_n centered at the origin with radii $r_n \longrightarrow \infty, \quad r_1 \le r_2 \le \ldots$.

Set

$$\alpha_n := \max_{\lambda \in C_n} \|QR_\lambda(A)\|_{B\to B}$$

and

$$\beta_n := \max_{\lambda\in C_n} \sup_{f\in B,\ f\neq 0} \|R_\lambda(A)f\|_D / \|f\|_B.$$

In [74] N.P.Kuptsov established the following general theorem.

Theorem 7.1 (N.P.Kuptsov). *Assume that $\alpha_n \longrightarrow 0, \quad n \longrightarrow \infty$. Then there exists a natural N such that for $n \ge N \quad \exists R_\lambda(A+Q)$.*

If in addition $\alpha_n \longrightarrow 0$ and $\beta_n\alpha_n = O(1/r_n)$, then for any $f \in B$

$$\left\| \frac{1}{2\pi i}\int_{C_n} (R_\lambda(A+Q)f - R_\lambda(A)f)\, d\lambda \right\|_D = o(1), \qquad n \longrightarrow \infty.$$

Take for instance $A = D^n$ in $[0,1]$ with regular two-point boundary conditions and let $B = L[0,1], \quad D = D_A, \quad \|y\|_D := \|y\|_{C[0,1]}$.

Set

$$Qy = \int_0^1 D^{n-2}y(t)d_t\sigma(x,t)$$

where $\underset{0}{\overset{1}{Var}}{}_x\ \sigma(x,t) =: q(x) \in L[0,1]$ and the theorem 7.1 applies.

The proof rests on the formula

$$-\frac{1}{2\pi i}\int_{C_n} \big(R_\lambda(A+Q)f - R_\lambda(A)f\big)d\lambda = \frac{1}{2\pi i}\int_{C_n} R_\lambda(A)QR_\lambda(A)fd\lambda + V_nf$$

with the remainder's estimate

$$\|V_nf\|_D = O\left(\alpha_n \cdot \|f\|_B\right),$$

which arises after iterating one time the identity connecting resolvents of the main and perturbed operators.

7.2 Commutator approach

In [86] and in subsequent papers we developped a new machinery in order to handle the equiconvergence problems. Below we shall illustrate it briefly in the simplest situation.

Let $D'(\mathbb{T})$ be the space of generalized functions on the one-dimensional torus

$$\mathbb{T} := \mathbb{R}\backslash\mathbb{Z}, \qquad D'(\mathbb{T}) = \left(C^\infty(\mathbb{T})\right)'.$$

For any $F \in D'(\mathbb{T})$ set $\hat F(l) := F(e_{-l}), \ \ e_l = \frac{1}{\sqrt{2\pi}}\exp(ilx)$,

$$S_r(F) := \sum_{|l|\le r/2\pi} \hat F(l)\cdot e_l.$$

Introduce a space $PF(\mathbb{T})$ of pseudofunctions on $\mathbb{T}$ as a subspace of $D'(\mathbb{T})$ with vanishing Fourier coefficients as $|l| \longrightarrow \infty$,

$$\|F\|_{PF} = \sup_l \left|\hat{F}(l)\right|.$$

Let K be a compact in $\mathbb{T}$. Introduce a seminorm in $C(K)$:

$$\|f\|_{A(K)} := \inf_{g|_K \equiv f} \|g\|_A .$$

Denote $A(K)$ the lineal in $C(K)$ with finite seminorm $\| \ \|_{A(K)}$.

Theorem 7.2 (A.M.Minkin, localization principle). *Given $F \in D'(\mathbb{T})$, $F|\Omega = 0$ for some open set $\Omega \subset \mathbb{T}$. Then*

$$\lim_{r\to\infty} \|S_r(F)\|_{A(K)} = 0$$

for any compact $K \subset \Omega$.

Its proof relies on a proposition which generalizes one theorem which goes back to A.Rajchman [6, p.194–196].

Lemma 7.3. *Let $\gamma \in C^1(\mathbb{T})$ be such that $\gamma' \in A$. Then for any $F \in PF$*

$$\lim_{r\to\infty} \|[S_r, \gamma](F)\|_A = 0,$$

where $[\ ,]$ stands for the operators' commutator.

Proof. Obviously,

$$\widehat{(\gamma \cdot F)}(l) = \sum_{j+k=l} \hat{F}(k) \cdot \hat{\gamma}(l).$$

Hence,

$$S_r(\gamma \cdot F) = \sum_{l,k\in P_1} \hat{F}(k)\hat{\gamma}(l-k) \cdot e_l(x),$$

where $P_1 = \{(l,k) \ | \ |l| \le r/2\pi, \ -\infty < k < \infty\}$. Conversely, expanding $\gamma(x)$ into an absolutely convergent trigonometric Fourier series we arrive at identity

$$\gamma \cdot S_r(F) = \sum_{l,k\in P_2} \hat{F}(k)\hat{\gamma}(l-k) \cdot e_l(x),$$

where $P_2 = \{(l,k) \ | \ |k| \le r/2\pi, \ -\infty < l < \infty\}$. Then

$$S_r(\gamma \cdot F) - \gamma \cdot S_r(F) = \sum_{(l_1,l_2)\in P_1\setminus P_2} - \sum_{(l_1,l_2)\in P_2\setminus P_1} . \tag{7.1}$$

Clearly, $P_1 \setminus P_2 = \{(l,k) \mid |l| \leq r/2\pi < k\}$ and $P_2 \setminus P_1 = \{(l,k) \mid |k| \leq r/2\pi < l\}$. But the number of integer points (l,k) lying on the line $l-k=p$ inside the domain of summation in the subtrahend or the minuend in (7.1) doesn't exceed p, whence

$$\begin{aligned} \left\| \sum_{P_1 \setminus P_2} \right\|_A &= \sum_{P_1 \setminus P_2} \left| \hat{F}(k) \right| \cdot |\hat{\gamma}(l-k)| \leq \\ \sum_{p=-\infty}^{\infty} |p| \cdot |\hat{\gamma}(p)| \cdot \|F\|_{PF} &= \|\gamma'\|_A \cdot \|F\|_{PF} \,. \end{aligned}$$

It suffices now to verify the lemma's assertion for trigonometric polynomials and then apply the Banach-Steinhaus theorem. □

Theorem 7.2 follows immediately after taking a smooth function γ which is identically 1 in some neighborhood of the compact K.

The theorem's statement seems to be new. Usually the Riemann summability to zero of a general trigonometric series is required. However, convergence in the sense of generalized functions is more flexible and better suits for applications in the theory of differential operators. Note also that the $A(K)-$ convergence is stronger than the $C(K)-$ one.

7.3 F.Schäfke's approach

In the beginning of sixties F.Schäfke developped a general equiconvergence principle and applied it to convergence of e.f. expansions in the complex domain. However we didn't succeed in translating his general constructions to the case of an interval of thev real axis and hence were unable to provide a comparison with results of other researchers. Therefore to our regret we can only refer the reader to his three thorough articles [110].

Chapter 2

Equiconvergence on the whole interval

1 Introduction

1.1 Notations

Let $n = 2q$ and consider two Birkhoff-regular n-th order differential operators L_1 and L_2 defined by two b.v.p. like (1.1.1)-(1.1.2). In what follows set $r_k = 2\pi + \alpha, k = 1, 2, \dots$ Then under an appropriate choice of $\alpha > 0$ condition (1.1.15) is fulfilled for sets of ch.v. of both operators L_1, L_2. In the sequel $\| \, \|$ stands for $C(0,1)$-norm and $\| \, \|_{(a,b)}$ for the norm in $C(a,b)$.

1.2 Order two case

At first, note that propositions 1.1.3, 1.1.5 don't solve the problem of equiconvergence on the whole interval $[0,1]$. And it is evident that in this case some additional restrictions should be imposed on the expanded function. Of course, it is possible to require more and more delicate smoothness conditions but this process seems to be infinite without any hope for a final solution. The situation completely changed in 1975 when A.P.Khromov made a decisive breakthrough [56]. His main idea was to *reduce* the equiconvergence problem for general e.f. expansions to that of some model function system, namely, to the question of *uniform convergence of a trigonometric series*. Let us state his result. Its formulation is slightly modified but a proof we found is even simpler than the original one and is given below in the section 6

Theorem 1.1 (Khromov). *Given two Birkhoff–regular second order differential operators L_1, L_2 and a function $f \in C(0,1)$, assume that*

$$f \in clos(D_{L_1}) \cap clos(D_{L_2}). \tag{1.1}$$

The closure is taken in $C(0,1)$. Then

$$\lim_{k\to\infty} \|S_{r_k}(f,L_1) - S_{r_k}(f,L_2)\| = 0 \tag{1.2}$$

if and only if

$$\lim_{r\to\infty} \|\sigma_r(\Phi_0)\|_{(-\delta,0)} = \lim_{r\to\infty} \|\sigma_r(\Phi_1)\|_{(-\delta,0)} \tag{1.3}$$

for some auxiliary functions Φ_0, Φ_1 and any fixed δ, $0 < \delta < \frac{1}{2}$.

Remark 1.2. More precisely Φ_0, Φ_1 are linear combinations of f, $f^{\#}$, where

$$f^{\#}(\xi) := f(1-\xi), \quad 0 \le \xi \le 1,$$

$\Phi_0, \Phi_1 \equiv 0$ off $(0,1)$ and they are explicitly defined in (1.8).

Thus the problem which is set above can be reduced to the classical question of trigonometric series convergence. We recall that there are a lot of strong convergence criteria for these serii which go back to Young, Lebesgue, de la Vallée-Poussin and others (see [136]). The latest belongs to E.Wermuth [134] and generalizes all the previous ones.

1.3 X-equivalence

In his thesis E.Wermuth [133, p.61-73] raised an equiconvergence problem on the whole interval for two e.f. expansions associated with Birkhoff-regular operators L_1, L_2 *simultaneously for all functions* in some class X. More precisely, he introduced the following

Definition 1.3. Given two n-th order differential operators $L_1, L_2 \in (R)$ in $L^2(0,1)$ and a function class $X \subset L^1(0,1)$ we say that these operators are X-equivalent if

$$\lim \|S_{r_k}(f,L_1) - S_{r_k}(f,L_2)\| = 0 \ \ \forall f \in X.$$

We shall also say that operators L_1, L_2 **essentially coincide** if the orders r_j and the leading parts V_J of their boundary forms are identical. Then E.Wermuth's theorem [133, Satz 13] states that two Birkhoff-regular operators L_1, L_2 essentially coincide if and only if they are $L^1(0,1)$-equivalent.

This statement is also valid for $X = L^p(0,1)$, where p is a fixed number, $1 \le p < \infty$ (see [133, pp.63,72]). The less can be chosen the set X, the weaker is the theorem's claim and the stronger is the result.

The main difficulty in E.Wermuth's problem consists of finding necessary and sufficient conditions for uniform boundedness of the family of operators:

$$f \to (S_{r_k}(f,L_1) - S_{r_k}(f,L_2)), \quad k = 1,2,3\ldots,$$

acting from X to $C[0,1]$. He solved it for $X = L^p(0,1)$, $1 \le p < \infty$. The extreme case $p = \infty$ and all the more $X = C(0,1)$ remained open [133, p.73]. Of course, this result gives no answer to equiconvergence on the whole interval for any *given fixed function* f as would be desired if we intend to generalize the Tamarkin-Stone's theorem 1.1.3. Obviously, this is a much more subtle question than the analogous one for a class of functions.

1.4 Higher order case

Hence, in sharp contrast with the case of equiconvergence in the internal points there is *absolutely no results* concerning generalization of the Tamarkin-Stone's theorem to the whole interval when $n > 2$. Such state of affairs stems from the dificulty of the problem in question. The standard resolvent's approach is good enough to obtain *sufficient* conditions provided f is enough smooth but fails to give necessary and sufficient ones, see, for instance [50, 51, 24] and others (the list may be considerably increased).

At first, we give such criterion provided

$$f = f_0 \in C_0(0,1) := \{g \in C(0,1):\ g(0) = g(1) = 0\} \tag{1.4}$$

and n is even, see theorems 1.4, 4.3. A more general situation with $f \in C(0,1)$ is reduced to this one in the section 5. Odd order case is solved in the section 7.

To begin with let us briefly sketch the idea of the proof. We have a quantity (namely, a difference of two partial sums) which tends to zero in $C(0,1)$. Expand then the numerator of the Green function along the uppermost row and each of the occurring minors with respect to the leftmost column, containing the column-vector W (if any). In the appearing finite sum some summands tend to zero uniformly on $[0,1]$. After eliminating them we obtain an equation with a sum of some leading terms from the left and $o(1)$ from the right. In order to extract these terms we need, say, a system of linear equations with a right-hand side $o(1)$. But we have only one equation with many unknowns!

Here a new idea is invoked: *we differentiate the left-hand side j times* and divide the result by r^j, $j = 0, \ldots, n$. Then each of the leading terms is factored by ε_k^j (see below) in the jth equation up to additional summands of the type $o(1)$. When $j = n$ we return to the original equation. Of course, the right-hand side is changed but it remains $o(1)$ and its concrete value is of no importance for us. Next we employ another important idea, we invoke *inequality for derivatives* and thus arrive at the desired system (after joining some terms pairwise but we omit details).

Further we shall need some notations. Let $n = 2q > 2$. For any n-th order differential operator $L \in (R)$ let A_j, B_j be n-columns,

$$A_j := [a_\nu \varepsilon_j^{\sigma_\nu}]_{\nu=0}^{n-1}, \quad B_j := [b_\nu \varepsilon_j^{\sigma_\nu}]_{\nu=0}^{n-1}, \quad j = 0, \ldots, n-1, \tag{1.5}$$

the regularity determinant $\Theta = \Theta(b^0, b^1) \neq 0$ reads as follows

$$\Theta := \det[A_0 \ldots A_{q-1} B_q \ldots B_{n-1}], \tag{1.6}$$

$\Theta^{\nu k}$ be a cofactor of the (ν, k) entry of Θ, indices ν, k vary from 0 to $n-1$. The same notation will be used for other matrices. Let

$$d_{m\nu} = \begin{cases} -a_\nu, & q \le m \le n-1 \\ \ \ b_\nu, & 0 \le m \le q-1 \end{cases}$$

and set

$$\alpha_{mk} = \alpha_{mk}(L) = \frac{1}{2\pi} \sum_{\nu=0}^{n-1} \varepsilon_m^{-(n-1-\sigma_\nu)} d_{m\nu} \cdot \Theta^{\nu_k}/\Theta. \tag{1.7}$$

For $f \in C(0,1)$ introduce a collection of functions associated with the given function f:

$$\Phi_k(\xi, f, L) := \alpha_{qk} f(\xi) + \alpha_{0k} f^{\#}(\xi), \qquad k = 0, \dots, n-1; \tag{1.8}$$
$$\Psi_k(\xi, f, L) := \alpha_{n-1,k} f(\xi) + \alpha_{q-1,k} f^{\#}(\xi), \qquad k = 0, \dots, n-1. \tag{1.9}$$

For the sake of brevity we shall write further r, Γ_r instead of r_k and Γ_{r_k}, respectively.

Theorem 1.4. *Given a function f_0 satisfying (1.4) and assume that*

$$span(\varphi_0, \varphi_q) = span(\psi_q, \psi_{n-1}) = span(f_0, f_0^{\#}) \tag{1.10}$$

where

$$\begin{cases} \varphi_k := \Phi_k(\cdot, f_0, L_1) - \Phi_k(\cdot, f_0, L_2), \\ \psi_k := \Psi_k(\cdot, f_0, L_1) - \Psi_k(\cdot, f_0, L_2), \end{cases} \qquad k = 0, \dots, n-1. \tag{1.11}$$

Let

$$I_r^{\pm}(f_0)(x) := \int_{1/r}^{1} (x+\xi)^{-1} \exp(\pm ir\xi) f_0(\xi)\, d\xi. \tag{1.12}$$

Then

$$\lim_{r\to\infty} \|S_r(f_0, L_1) - S_r(f_0, L_2)\| = 0 \tag{1.13}$$

if and only if

$$\lim_{r\to\infty} \|I_r^{\pm}(g)\| = 0, \quad g = f_0, f_0^{\#}. \tag{1.14}$$

Let us indicate that in theorem 1.4 only *a unique function* f_0 is considered, i.e. we merely put a set $X = \{f_0\}$ consisting of *one* element. Hence in our case the operators L_1, L_2 may be absolutely different and really we have established a *true generalization* of the Tamarkin-Stone's Tamarkin- theorem.

Remark 1.5. It is possible to change $\| \ \|$ in (1.14) by $\| \ \|_{(0,\delta)}$ for any fixed $\delta, 0 < \delta < 1$. There is also a simple sufficient condition for (1.10) to be valid:

$$\det \begin{bmatrix} \beta_{00} & \beta_{q0} \\ \beta_{0q} & \beta_{qq} \end{bmatrix} \neq 0, \quad \det \begin{bmatrix} \beta_{q-1,q-1} & \beta_{n-1,q} \\ \beta_{q-1,n-1} & \beta_{n-1,n-1} \end{bmatrix} \neq 0, \tag{1.15}$$

where

$$\beta_{mk} := \alpha_{mk}(L_1) - \alpha_{mk}(L_2).$$

Of course, it is equivalent to (1.10) if the functions f_0 and $f_0^{\#}$ are linearly independent.

2 Green's function

2.1 New fundamental system of solutions

Consider a Birkhoff-regular b.v.p.

$$D^n y = \lambda y + f, \tag{2.1}$$

$$V_\nu(y) = 0, \quad \nu = 0, \ldots, n-1. \tag{2.2}$$

Equation $D^n y = \lambda y$ has an evident f.s.s. , namely

$$y_k(x, \varrho) \equiv \exp(i\varrho\varepsilon_k x), \quad k = 0, \ldots, n-1. \tag{2.3}$$

Introducing kernels

$$g(x, \xi, \varrho) = i \cdot \begin{cases} \sum\limits_{k=0}^{q-1} \varepsilon_k^{-(n-1)} y_k(x-\xi), & x > \xi \\ -\sum\limits_{k=q}^{n-1} \varepsilon_k^{-(n-1)} y_k(x-\xi), & x > \xi \end{cases} \tag{2.4}$$

and

$$g_0(x, \xi, \varrho) := g(x, \xi, \varrho)/(n\varrho^{n-1}) \tag{2.5}$$

we get a particular solution $g_0(f)$ of (2.1),

$$g_0(f) := \int_0^1 g_0(x, \xi, \varrho) f(\xi)) d\xi.$$

In the sequel it will be more convenient to use another f.s.s. $\{z_k\}_{k=0}^{n-1}$, where

$$z_k(x, \varrho) := \begin{cases} y_k(x, \varrho), & k = 0, \ldots, q-1, \\ y_k(x-1, \varrho), & k = q, \ldots, n-1. \end{cases} \tag{2.6}$$

This choice of a f.s.s. is natural due to the fact that

$$z_k = \mathrm{O}(1)\ , k = 0, \ldots, n-1; \quad g(x, \xi, \varrho) = \mathrm{O}(1), \ \ 0 \le x, \xi \le 1 \tag{2.7}$$

for $\varrho \in S_0 := \{0 \le \arg\varrho \le 2\pi/n\}$.

2.2 Green's function representation

By variation of constants we get a well-known expression (1.2.1)-(1.2.2) for the Green's function as a ratio of two determinants with the f.s.s. being chosen as in (2.6). Further

on, canceling powers of ϱ in the nominator and denominator, we get problem's (2.1)–(2.2) solution of the form :

$$y = G(f) := \int_0^1 G(x, \xi, \varrho) f(\xi)\, d\xi,$$

where

$$G(x, \xi, \varrho) = i \cdot \det H / (n\varrho^{n-1} \det \eta). \tag{2.8}$$

Here

$$\eta = [\eta_{\nu k}]_0^{n-1}, \quad \eta_{\nu k} := \varepsilon_k^{\sigma_\nu} (b_\nu z_k(1) + a_\nu z_k(0)), \tag{2.9}$$

$$H(x, \xi, \varrho) := \begin{bmatrix} g & z^T \\ W & \eta \end{bmatrix}, \tag{2.10}$$

square brackets here denote a matrix and z^T stands for the transposed of the column-vector $z = (z_k(x, \varrho))_0^{n-1}$,

$$W = (W_\nu)_{\nu=0}^{n-1}, \quad W_\nu = \sum_{m=0}^{n-1} \varepsilon_m^{-(n-1-\sigma_\nu)} d_{m\nu} u_m(\xi, \varrho) \tag{2.11}$$

and at last

$$u_m(\xi, \varrho) := \begin{cases} y_m(1 - \xi, \varrho), & m = 0, \dots, q-1, \quad 0 \le \xi \le 1 \\ y_m(-\xi, \varrho), & m = q, \dots, n-1. \end{cases} \tag{2.12}$$

Changing a little bit notation from [19, p.1185] we use an abbreviation:

$$[[q]] := q + O(e^{-\delta|Im\varrho|}) + O(e^{-\delta|\varepsilon_{q-1} Im\varrho|}) + O(\frac{1}{\varrho}), \quad \varrho \in \Gamma_r.$$

The quantity q may vary with ϱ.

Lemma 2.1. *The following relation is valid*

$$(\det \eta)^{-1} = [[\Theta^{-1}]]. \tag{2.13}$$

Proof. We have that

$$z_k(0) = [[0]], \quad k = q, \dots, n-1; \quad z_k(1) = [[0]], \quad k = 0, \dots, q-1. \tag{2.14}$$

Let us expand $\det \eta$ into a sum of determinants with only $z_k(0)$ or $z_k(1)$ in each column. Then all of them become $[[0]]$ except one which coincides with Θ. Moreover, from [97, p.77–78] it follows that

$$|\det \eta| \ge C, \quad \varrho \in S_\delta := S \setminus \{|\varrho - \varrho_j^i| \le \delta\}_{j=1}^\infty, \quad i = 1, 2 \tag{2.15}$$

for sufficiently small $\delta > 0$. Here ϱ_j^i stands for the ch.v. of the operator L_i, $i = 1, 2$. Therefore,

$$(\det \eta)^{-1} - \Theta^{-1} = (\Theta - \det \eta)/(\Theta \det \eta) = [[0]]. \tag{2.16}$$

$\square$

A similar calculation can be found in [19, p.1186].

2.3 A partial sum's formula

Let us consider a partial sum

$$S_r(f) := (-2\pi i)^{-1} \int_{\Gamma_r} G(f)(x, \varrho) n \varrho^{n-1} \, d\varrho \tag{2.17}$$

of e.f. expansion for the b.v.p. (2.1)–(2.2). Taking into account (2.8),(2.14) and expanding $\det H$ with respect to the first row and each of the appearing minors except the first one with respect to the column W we arrive at identity

$$S_r(f) \equiv S_{r,0}(f) + \sum_{k,\nu,m=0}^{n-1} J_{r,m,\nu,k}(f), \tag{2.18}$$

where

$$S_{r,0}(f) := (-2\pi i)^{-1} \int_{\Gamma_r} g(f) \, d\varrho, \tag{2.19}$$

$$g(f) := \int_0^1 g(x, \xi, \varrho) f(\xi) \, d\xi$$

and

$$\begin{aligned} J_{r,m,\nu,k}(f) := \quad & (-2\pi)^{-1} \textstyle\int_{\Gamma_r} \{ z_k(x, \varrho)(-1)^{k+3+k}([[\Theta^{\nu k}]]/\Theta) \\ & \cdot \left\{ \int_0^1 f(\xi) u_m(\xi, \varrho) \, d\xi \varepsilon_m^{-(n-1-\sigma_\nu)} \, d_{m\nu} \right\} d\varrho. \end{aligned} \tag{2.20}$$

Here we also took into account the identity $\eta^{\nu k} \equiv [[\Theta^{\nu k}]]$. It can be easily deduced expanding $\det H$ along the uppermost row and each of the occurring minors along the leftmost column.

It will be helpful to rewrite (2.20) in the following way:

$$J_{r,m,\nu,k}(f) = \alpha_{m\nu k} \cdot \tau_{rmk}(f[[1]]) \tag{2.21}$$

where

$$\alpha_{m\nu k} := (2\pi)^{-1} \varepsilon_m^{-(n-1-\sigma_\nu)} d_{m\nu} (\Theta^{\nu k}/\Theta) \tag{2.22}$$

and

$$\tau_{rmk}(f) := \int_{\Gamma_r} z_k(x,\varrho) \left(\int_0^1 f(\xi) u_m(\xi,\varrho) \right) d\varrho. \tag{2.23}$$

Remark 2.2. By an elementary computation (see also analogous result in [19, p.1191]) we get that

$$S_{r,0}(f) \equiv \sigma_r(f). \tag{2.24}$$

3 Equiconvergence with a trigonometric Fourier integral

3.1 Simplifications

Lemma 3.1. *Let us represent the second factor in (2.21) as a sum*

$$\tau_{rmk}(f) + \tau_{rmk}(f[[0]]). \tag{3.1}$$

Then the following estimate is valid

$$\|\tau_{rmk}(f[[0]])\| = o(1) \ \textit{as} \ \ r \to \infty, \ f \in L(0,1). \tag{3.2}$$

Proof. We shall use a well-known relation:

$$\sup_{r>0} \int_{\Gamma_r} \exp(-\varepsilon|Im\varrho|)|\,d\varrho| < \infty. \tag{3.3}$$

Thus we have a uniformly bounded family of linear operators acting from $L(0,1)$ to $C(0,1)$:

$$f \to \tau_{rmk}(f[[0]]). \tag{3.4}$$

Obviously it tends to zero on a dense lineal $C_0^\infty(0,1)$ since the factor $[[0]]$ depends only on ϱ, not on ξ. Therefore it remains to apply the Banach-Steinhaus theorem. ☐

Lemma 3.2. *For any* $m \notin \{0, q-1, q, n-1\}$ $\quad \|\tau_{rmk}(f)\| = o(1)$ *as* $r \to \infty$.

Proof. Assume for defficiency that $0 < m < q-1$. Then

$$\begin{aligned} |u_m(\xi,\varrho)| &= |y_m(1-\xi,\varrho)| \\ &= \exp\bigl(-Im(\varrho\varepsilon_m)(1-\xi)\bigr) \\ &\le \exp\bigl(-\varepsilon|\varrho|(1-\xi)\bigr) \end{aligned}$$

with some positive ε. Hence

$$\int_{\Gamma_r}\int_0^1 |u_m(\xi,\varrho)|\,d\xi\,|d\varrho| \le \int_{\Gamma_r} |\,d\varrho| \int_0^1 exp\{-\varepsilon|\varrho|(1-\xi)\}\,d\xi \tag{3.5}$$

$$= (2\pi/n)\int_0^1 r\exp\{-\varepsilon r(1-\xi)\}\,d\xi \le \varepsilon^{-1}2\pi/n.$$

Since $f \in C_0(0,1)$, it suffices to recall (2.7) and apply the Banach–Steinhaus theorem to the family $r \to \tau_{rmk}(f)$ of operators acting from $C_0(0,1)$ to $C(0,1)$. $\square$

3.2 Remainder formula

Now we consider the sum

$$\sum_{k,\nu,m=0}^{n-1} J_{r,m,\nu,k}(f) \tag{3.6}$$

and join pairwise all its summands with $m=0$ and $m=q$ or $m=q-1$ and $m=n-1$, respectively. Lemmas 3.1, 3.2 together with (2.19), (2.22) yield an important identity

$$\begin{aligned} S_r(f)-\sigma_r(f) \equiv{}& error \\ &+\sum_{k=0}^{n-1}\big[\ \{\alpha_{0k}\tau_{r0k(f)}+\alpha_{qk}\tau_{rqk}(f)\} \\ &\qquad +\{\alpha_{q-1,k}\tau_{r,q-1,k}(f)+\alpha_{n-1,k}\tau_{r,n-1,k}(f)\}\big], \end{aligned} \tag{3.7}$$

where *error* stands for the summands tending to zero in $C(0,1)$ as $r\to\infty$.

Lemma 3.3. *The following identities are valid*

$$\alpha_{0k}\int_0^1 f(\xi)u_0(\xi,\varrho)\,d\xi + \alpha_{qk}\int_0^1 f(\xi)u_q(\xi,\varrho)\,d\xi$$

$$\equiv \int_0^1 \Phi_k(\xi)y_0(\xi,\varrho)d\xi, \tag{3.8}$$

$$\alpha_{q-1,k}\int_0^1 f(\xi)u_{q-1}(\xi,\varrho)\,d\xi + \alpha_{n-1,k}\int_0^1 f(\xi)u_{n-1}(\xi,\varrho)\,d\xi$$

$$\equiv \int_0^1 \Psi_k(\xi)y_{q-1}(\xi,\varrho)d\xi, \tag{3.9}$$

Proof. It suffices to notice that $y_{j+q}(\xi,\varrho) \equiv y_j(-\xi,\varrho)$ and to make a substitution: $\xi \to 1-\xi$ if needed. $\square$

Corollary 3.4. *Formulas (3.7)–(3.9) yield a final relation:*

$$\begin{aligned} S_r(f) - \sigma_r(f) \equiv\ & error \\ & + \sum_{k=0}^{n-1}\Big\{\int_{\Gamma_r} z_k(x,\varrho)\int_0^1 \Phi_k(\xi) y_0(\xi,\varrho) d\xi\, d\varrho \\ & + \int_{\Gamma_r} z_k(x,\varrho)\int_0^1 \Psi_k(\xi) y_{q-1}(\xi,\varrho) d\xi\, d\varrho\Big\} \\ & = error + \sum_{k=0}^{n-1}\{\eta_k(x,r,f) + \zeta_k(x,r,f)\}, \end{aligned} \tag{3.10}$$

where $\|error\| \to 0$ *as* $r \to \infty$.

3.3 Preliminary transformations

Set

$$\gamma_0 = \gamma_0(x,r,f) := \sum_{k=0}^{q-1}[\eta_k(x,r,f) + \zeta_k(x,r,f)], \tag{3.11}$$

$$\gamma_1 = \gamma_1(x,r,f) := \sum_{k=q}^{n-1}[\eta_k(x,r,f) + \zeta_k(x,r,f)]. \tag{3.12}$$

Then

$$S_r(f) - \sigma_r(f) = \gamma_0 + \gamma_1 + error.$$

Lemma 3.5. *An estimate*

$$\|S_r(f) - \sigma_r(f)\| = o(1), \quad r \to \infty$$

is valid if and only if for any fixed δ, $\quad 0 < \delta < 1/2$

$$\|\gamma_0\|_{(0,\delta)} = o(1), \quad \|\gamma_1\|_{(1-\delta,1)} = o(1) \ \text{ as } r \to \infty. \tag{3.13}$$

Proof. It suffices to show that

$$\|\eta_k\|_{(\delta,1)} = o(1), \qquad \|\zeta_k\|_{(\delta,1)} = o(1), \qquad k = 0,\dots,q-1, \tag{3.14}$$

$$\|\eta_j\|_{(0,1-\delta)} = o(1), \qquad \|\zeta_j\|_{(0,1-\delta)} = o(1), \qquad j = q,\dots,n-1 \tag{3.15}$$

as $r \to \infty$. To be definite we shall consider only (3.14). But

$$\|z_k(\cdot,\varrho)\|_{(\delta,1)} = O(\exp(-Im(\varrho\varepsilon_k)\delta)) = O(\exp(-\delta_1 Im\varrho)$$

for $\varrho \in \Gamma_r, \quad k = 0, \ldots, q-2$ and some positive δ_1. Quite analogously,

$$\|z_k(\cdot, \varrho)\|_{(\delta,1)} = O(exp(-\delta_1 Im\varrho))$$

for

$$\varrho \in \varepsilon_{q-1}\Gamma_r := \{\varrho\varepsilon_{q-1} : \varrho \in \Gamma_r\}.$$

The circular arc $\varepsilon_{q-1}\Gamma_r$ lies in the upper half-plane. Therefore, we can use (3.3) in any case. At last the same hint with the Banach-Steinhaus theorem completes the proof. □

3.4 Behaviour of the main terms under differentiation

Let us consider a difference

$$D^n\eta_k - r^n\eta_k = \int_{\Gamma_r} (\varrho^n - r^n) z_k(x, \varrho) \int_0^1 \Phi_k(\xi) y_0(\xi) d\xi \, d\varrho. \tag{3.16}$$

Lemma 3.6. *The following relation holds uniformly with respect to x, $0 \le x \le 1$:*

$$D^j\eta_k - (r\varepsilon_k)^j\eta_k = o(r^j), \quad j = 1, \ldots, n; \quad k = 0, \ldots, n-1. \tag{3.17}$$

Proof. Since

$$D^j z_k(x, \varrho) \equiv (\varrho\varepsilon_k)^j z_k(x, \varrho)$$

we see that the left-hand side in (3.17) differs from η_k by the additional factor $\varepsilon_k^j(\varrho^j - r^j)$ in (3.10). Then, taking into account (2.7) we get that

$$\|D^j\eta_k - (r\varepsilon_k)^j\eta_k\| \le C \int_{\Gamma_r} |\varrho^j - r^j| \int_0^1 |y_0(\xi)| \, d\xi \, |d\varrho| \cdot \|f\|. \tag{3.18}$$

In the meantime,

$$|y_0(\xi, \varrho)| \equiv \exp(-Im\varrho\, \xi) \le \exp(-2r\varphi\xi/\pi),$$

$\varrho = r\exp(i\varphi), \quad 0 \le \varphi < 2\pi/n$. Then the right-hand side in (3.18) is less or equal to

$$\begin{aligned} &C\|f\|r^{j+1} \int_0^{2\pi/n} d\varphi \int_0^1 d\xi \exp(-2r\varphi\xi/\pi)|\exp(ij\varphi) - 1| \\ &\le C\|f\|r^{j+1} \int_0^{2\pi/n} d\varphi \int_0^1 d\xi \, \exp(-2r\varphi\xi/\pi) O(j\varphi). \end{aligned} \tag{3.19}$$

Replacing the internal integral in (3.19) by

$$\int_0^\infty j\varphi \exp(-2r\varphi\xi/\pi) = \pi j/2r$$

we come to (3.17) with $O(r^j)$ instead of $o(r^j)$. It remains now to apply the Banach-Steinhaus theorem. □

Remark 3.7. Proceeding in the same way we find that

$$D^j \zeta_k - (r\varepsilon_1\varepsilon_k)^j \zeta_k = o(r^j), \quad j = 1, \dots, n; \quad k = 0, \dots, n-1$$

as $r \to \infty$. In this case we have only to take into account that the modulus

$$|y_{q-1}(\xi, \varrho)|, \quad \varrho \in \Gamma_r$$

attains its maximum at the point $r\varepsilon_1$, not at r as $|y_0(\xi, \varrho)|$.

Lemma 3.8. *Let δ be any fixed number, $0 < \delta < 1/2$. Then the following inequalities hold for $r \geq 1$ and $j = 1, \dots, n$*

$$\|D^j \gamma_0\|_{(0,\delta)} \leq C r^j \|\gamma_0\|_{(0,\delta)} + o(r^j), \tag{3.20}$$

$$\|D^j \gamma_1\|_{(1-\delta,1)} \leq C r^j \|\gamma_1\|_{(1-\delta,1)} + o(r^j). \tag{3.21}$$

Proof. For the sake of being definite consider only (3.20). According to lemma 3.6 and remark 3.7 we have that

$$D^j \gamma_0 = \sum_{k=0}^{q-1} \{(r\varepsilon_k)^j \eta_k + (r\varepsilon_1\varepsilon_k)^j \zeta_k\} + o(r^j), \quad j \geq 1 \tag{3.22}$$

uniformly with respect to x, $0 \leq x \leq \delta$. Thus for $j = n$ we have that

$$D^n \gamma_0 = r^n \gamma_0 + o(r^n), \quad 0 \leq x \leq \delta. \tag{3.23}$$

Now it is time to recall an inequality for derivatives [10, p.131]. Let g be any continuously differentiable function in $(0, \delta)$. Then

$$\|g^{(j)}\|_{(0,\delta)} \leq C_1 (\|g\|_{(0,\delta)})^{(n-j)/n} (\|g^{(n)}\|_{(0,\delta)})^{j/n} + C_2 \|g\|_{(0,\delta)}.$$

Applying it to $g = \gamma_0|_{(0,\delta)}$ and taking into account (3.23) we come to an inequality

$$\|D^j \gamma_0\|_{(0,\delta)} \leq C_1 (\|\gamma_0\|_{(0,\delta)})^{(n-j)/n} (\|\gamma_0^n\|_{(0,\delta)})^j / n + C_2 \|\gamma_0\|_{(0,\delta)}.$$

This yields (3.20) after considering two cases:

$$i) \|\gamma_0\|_{(0,\delta)} \leq \varepsilon, \quad ii) \|\gamma_0\|_{(0,\delta)} \geq \varepsilon,$$

ε being sufficiently small. □

3.5 Main criterion

According to lemmas 3.6 and 3.8 we have that

$$\|S_r(f) - \sigma_r(f)\| = o(1)$$

if and only if

$$i)\ \ r^{-j}\|D^j\gamma_0\|_{(0,\delta)} = o(1), \qquad j = 0,\dots,q; \tag{3.24}$$

$$ii)\ r^{-j}\|D^j\gamma_1\|_{(1-\delta,1)} = o(1), \qquad j = 0,\dots,q. \tag{3.25}$$

Of course, we can replace the number q in (3.24)-(3.25) by any other nonnegative one but our choice is suitable for further purposes.

Substituting (3.22) into (3.24) we obtain a system of equations with respect to the variables η_k, ζ_k

$$\left\{\sum_{k=0}^{q-1}(\varepsilon_k^j\eta_k + \varepsilon_k^j\zeta_k) = o(1), \quad 0 \le x \le \delta, \quad j = 0,\dots,q\right.$$

or in a modified form

$$\left\{\begin{aligned} \varepsilon_0^j\eta_0 + \varepsilon_1^j(\eta_1+\zeta_0) + \dots \\ +\varepsilon_{q-1}^j(\eta_{q-1}+\zeta_{q-2}) + +\varepsilon_q^j\zeta_{q-1} = o(1), \\ j = 0,\dots,q, \quad 0 \le x \le \delta. \end{aligned}\right. \tag{3.26}$$

Since the system's determinant coincides with a Vandermonde one

$$|\varepsilon_k^j|_{j,k=0}^q,$$

we solve it immediately:

$$\eta_0 = o(1), \quad \eta_1 + \zeta_0 = o(1), \quad \dots, \eta_{q-1} + \zeta_{q-2} = o(1) \tag{3.27}$$

uniformly with respect to x, $\ 0 \le x \le \delta$. Proceeding in the same way, we derive from (3.25) that

$$\eta_q = o(1),\ \eta_{q+1} + \zeta_q = o(1), \dots, \eta_{n-1} + \zeta_{n-2} = o(1),\ \zeta_{n-1} = o(1) \tag{3.28}$$

uniformly with respect to x, $\ 1-\delta \le x \le 1$.

Theorem 3.9. *Let L be a Birkhoff-reqular differential operator of the form (1.1.1)-(1.1.2), δ be any positive number ≤ 1 and $f_0 \in C_0(0,1)$. Then*

$$\lim_{r\to\infty}\|S_r(f_0) - \sigma_r(f_0)\| = 0$$

if and only if relations (3.27)-(3.28) hold.

Proof. Use (3.27)–(3.28) and take into account (3.14)–(3.15). □

Corollary 3.10. *All variables in (3.27) are $o(1)$ for $\delta \le x \le 1$. So we can replace δ in (3.27) by 1 and similarly by 0 in (3.28).*

4 Modification of the criterion

4.1 Preliminary transformations

It is difficult to use theorem 3.9 directly. Therefore in this section we simplify its hypotheses.

Lemma 4.1. *Expressions* $\|\eta_0\|$, $\|\eta_q\|$, $\|\zeta_{q-1}\|$ *and* $\|\zeta_{n-1}\|$ *tend to zero as* $r \to \infty$ *if and only if the same is true for*

$$\|I_r^+(\varphi_0)\|,\ \|I_r^+(\varphi_q)\|,\ \|I_r^-(\Psi_{q-1})\| \text{ and } \|I_r^-(\Psi_{n-1})\|,$$

respectively.

Proof. Recall that

$$\eta_k = \int\limits_0^1 P_k(x,\sigma,r)\Phi_k(\xi)\,d\xi,\quad \zeta_k = \int\limits_0^1 Q_k(x,\sigma,r)\Psi_k(\xi)\,d\xi, \tag{4.1}$$

where

$$P_k(x,\sigma,r) = \int_{\Gamma_r} z_k(x,\varrho)y_0(\xi,\varrho)\,d\varrho, \tag{4.2}$$

$$Q_k(x,\sigma,r) = \int_{\Gamma_r} z_k(x,\varrho)y_{q-1}(\xi,\varrho)\,d\varrho. \tag{4.3}$$

All the four cases in the lemma can be proved in one and the same way. Therefore we will consider only the quantity $\|\eta_0\|$. Formulas (4.2) and (2.7) yield that

$$P_0(x,\xi,r) = O(r), \quad 0 \le x, \le 1.$$

Replace representation (4.1) for η_0 by the integral

$$\int\limits_{1/r}^1 P_0(x,\xi,r)\varphi_0(\xi)\,d\xi \tag{4.4}$$

with an error $o(1)$ as $r \to \infty$. Then a direct calculation shows that

$$P_0(x,\xi,r) = [exp(ir\varepsilon_1(x+\xi)) - exp(ir(x+\xi))]/[i(x+\xi)]. \tag{4.5}$$

Clearly

$$|exp(ir\varepsilon_1(x+\xi))| = exp(-rh(x+\xi))$$

with $h = Im\varepsilon_1 > 0$ and $(x+\xi)^{-1} \le r$ because $r^{-1} \le \xi \le 1,\ \ x \ge 0$. Since $\Phi_0 \in C_0(0,1)$ we have that

$$\|\int\limits_{1/r}^1 (x+\xi)^{-1}exp(ir\varepsilon_1(x+\xi))\Phi_0(\xi)d\xi\|$$

$$= O(r\int\limits_{1/r}^1 \exp(-rh\xi))|\Phi_0(\xi)|d\xi) = o(1).$$

Therefore, $\quad \|\eta_0\| = o(1)$ if and only if $\|I_r^+(\Phi_0)\| = o(1)$. ∎

4.2 Kernels' calculation

Let

$$u_{kp}(x,\xi) = \varepsilon_k x + \varepsilon_p \xi, \quad v_{kp}(x,\xi) = \varepsilon_k (x-1) + \varepsilon_p \xi. \tag{4.6}$$

Then

$$P_k(x,\xi,r) = (iu_{k0})^{-1}[\exp(ir\varepsilon_1 u_{k0}) - \exp(iru_{k0})] \tag{4.7}$$

for $k = 1, \dots, q-1$;

$$Q_k(x,\xi,r) = (iv_{k,q-1})^{-1}[\exp(ir\varepsilon_1 u_{k,q-1}) - \exp(iru_{k,q-1})] \tag{4.8}$$

for $k = 0, \dots, q-2$. Analogous formulas hold for

$$P_k, \quad k = q+1, \dots, n-1; \quad Q_k, \quad k = q, \dots, n-2$$

with v_{kp} instead of u_{kp} in the exponentials in square brackets. Factors $(\dots)^{-1}$ remain unchanged.

Lemma 4.2. *The following relations are valid*

$$\eta_k = i\exp(ir\varepsilon_k x) I_r^+(\Phi_k) + o(1), \qquad 1 \le k \le q-1; \tag{4.9}$$

$$\eta_k = i\exp(ir\varepsilon_k (x-1)) I_r^+(\Phi_k) + o(1), \qquad q+1 \le k \le n-1; \tag{4.10}$$

$$\zeta_j = (i\varepsilon_1)\exp(ir\varepsilon_{j+1} x) I_r^-(\Psi_j) + o(1), \qquad 0 \le j \le q-2; \tag{4.11}$$

$$\zeta_j = (i\varepsilon_1)\exp(ir\varepsilon_{j+1} (x-1)) I_r^-(\Psi_j) + o(1), \qquad q \le j \le n-2. \tag{4.12}$$

Proof. For the sake of brevity consider η_k for some k, $1 \le k \le -1$. Exponential factor in (4.9) is bounded. Then, repeating lemma's 4.1 proof, we get that

$$\eta_k = i\exp(ir\varepsilon_k x) \int_{1/r}^{1} (\varepsilon_k x + \xi)^{-1} \exp(ir\xi) \Phi_k(\xi) d\xi + o(1).$$

Taking into account an evident relation

$$(\varepsilon_k x + \xi)^{-1} - (x+\xi)^{-1} = \begin{cases} O(x^2/\xi), & \xi \ge x, \\ O(1/x), & \xi < x, \end{cases}$$

we come to (4.9). Formulas (4.10)–(4.12) can be attained in quite the same way. $\square$

4.3 Equiconvergence with a trigonometric series

We introduce now a nondegeneracy condition:

$$span(\Phi_0, \Phi_q) = span(\Psi_{q-1}, \Psi_{n-1}) = span(f_0, f_0^{\#}). \tag{4.13}$$

Then all the expressions below are $o(1)$,

$$\|I_r^+(\Phi_0)\|, \quad \|I_r^+(\Phi_q)\|, \quad \|I_r^-(\Psi_{q-1})\|, \quad \|I_r^-(\Psi_{n-1})\| = o(1) \tag{4.14}$$

if and only if (1.14) holds. Moreover, (4.14) yields that

$$\|\zeta_j\|, \|\eta_k\| = o(1) \quad \forall j, k.$$

It remains now to compare lemma 4.1 with theorem 3.9 and corollary 3.10 and we come to

Theorem 4.3. *Consiger a Birkhoff-reqular operator L defined by the b.v.p. (1.1.1)-(1.1.2). Let $f_0 \in C_0(0,1)$ and (4.13) be satisfied. Then*

$$\lim_{r\to\infty} \|S_r(f_0) - \sigma_r(f_0)\| = 0 \tag{4.15}$$

if and only if (1.14) is true.

4.4 End of theorem's 1.4 proof.

Applying (3.7) to the difference $S_r(f_0, L_1) - S_r(f_0, L_2)$ we get an analogous formula with new coefficients

$$\beta_{kj} := \alpha_{kj}(L_1) - \alpha_{kj}(L_2).$$

It suffices now to repeat theorem's 4.3 proof word by word, replacing (4.13) by (1.10).
□

5 Functions, satisfying zero-order conditions

Let now $n = 2q \geq 2$ and consider an n-th order operator $L \in (R)$. Let

$$f \in C[0,1], \quad f \in clos_{C[0,1]} D_L, \tag{5.1}$$

that is f satisfies zero-order normalized boundary conditions (if any). Set

$$P(x,f) = f(0)\cdot x + f(1)\cdot(1-x), \quad f_0(x) := f(x) - P(x,f), \tag{5.2}$$

$$\tilde{f}(x) = \begin{cases} f(0), & x < 0; \\ f(x), & 0 \leq x \leq 1; \\ f(1), & x > 1. \end{cases} \tag{5.3}$$

A direct calculation (we omit details) shows that

$$S_r(P) - \sigma_r(P) = f(0)\cdot\sigma_r(\chi_1)(x) + f(1)\cdot\sigma_r(\chi_2)(x) + o(1),$$

where χ_1 and χ_2 are characteristic functions of the intervals $(-\infty, 0)$ and $(1, \infty)$, respectively.

Hence, one obtains a refinement of the theorem 4.3 assuming validity of (5.1) and replacing (4.15) with

$$\lim_{r\to\infty} \|S_r(f) - \sigma_r(\tilde{f})\| = 0. \tag{5.4}$$

Quite analogously, theorem 1.4 is also true for continuous functions f, subject to (1.1) instead of (1.4). It is only needed to replace f_0 by f in (1.13).

6 Order two case

In this section we present a short proof of A.P.Khromov's theorem 1.1 when $f = f_0 \in C_0[0,1]$. The general case is covered by corollaries 6.1–6.2.

Proof. Repeating considerations from subsections 3.1–3.2 word by word we come to the formula (cf. (3.7))

$$\begin{aligned} S_r(f) - \sigma_r(f) &\equiv error \\ &+ \sum_{k=0}^{1} \big[\ \{\alpha_{0k}\tau_{r0k(f)} + \alpha_{1k}\tau_{r1k}(f)\} \\ &= error + \sum_{k=0}^{1} J_k, \end{aligned} \tag{6.1}$$

because for $n = 2$ $q-1 = 0$ and $q = n-1 = 1$ whereas the sum in (6.1) contains only two summands instead of four for $n = 2q > 2$. Then an analogue of lemma 3.3 gives

$$J_k = \tau_{r0k}(\Phi_k), \quad k = 0, 1.$$

Further, we can not argue as before differentiating the right-hand side of (6.1) since now the arc Γ_r is a semicircle with both endpoints on the real axis. Informally speaking these endpoints both affect an integral over $d\varrho$. Therefore it is not easy to evaluate the main part of such integral after differentiation.

Instead, we merely shrink the path of integration to the interval $[-r, r]$ and deduce that

$$J_0 = \int_{-r}^{r} e^{i\varrho x}\, d\varrho \cdot \int_0^1 \Phi_0(\xi) e^{i\varrho\xi}\, d\xi \tag{6.2}$$

$$J_1 = \int_{-r}^{r} e^{i\varrho(1-x)}\, d\varrho \cdot \int_0^1 \Phi_1(\xi) e^{i\varrho\xi}\, d\xi. \tag{6.3}$$

Extending both functions Φ_0, Φ_1 by zero off $[0,1]$ we readily obtain

$$J_0 = 2\pi \cdot \sigma_r(\Phi_0)(-x), \quad J_1 = 2\pi \cdot \sigma_r(\Phi_1)(x-1), \quad 0 \le x \le 1.$$

Then

$$\|S_r(f)-\sigma_r(f)\| \to 0 \Longleftrightarrow \|\sigma_r(\Phi_0)(-x)+\sigma_r(\Phi_1)(x-1)\| \to 0. \tag{6.4}$$

Fix any $\delta,\ 0<\delta<1$ and observe that

$$\max_{\delta\le x\le 1} \|\sigma_r(\Phi_0)(-x)\| \to 0, \quad \max_{0\le x\le 1-\delta} \|\sigma_r(\Phi_1)(x-1)\| \to 0 \tag{6.5}$$

according to the localization principle for trigonometric series. At last the proof completes by combining (6.4) and (6.5).

Corollary 6.1. *Incidentally we also established the following useful relation for a second order operator $L \in (R)$:*

$$S_r(f)(x)-\sigma_r(f)(x) = 2\pi\left(\sigma_r(\Phi_0)(-x)+\sigma_r(\Phi_1)(x-1)\right)+o(1), \quad 0\le x\le 1. \tag{6.6}$$

Corollary 6.2. *Given two second order operators $L_1, L_2 \in (R)$, the following relations are equivalent:*

$$1.\ \lim_{r\to\infty} \|S_r(f,L_1)-S_r(f,L_2)\| \to 0, \tag{6.7}$$

$$2.\ \lim_{r\to\infty} \max_{-\delta\le x\le 0} |\sigma_r(\varphi_k)(x)| = 0, \quad k=0,1, \tag{6.8}$$

provided that f obeys (1.1). Here φ_k are defined in (1.11), functions f and f_0 are related by (5.2), and the proof stems immediately from (6.6).

□

7 Odd order operators

In this section let m be an odd number, $m=2q+1$, $L\in(R)$ be an mth order differential operator in $L^2(0,1)$ defined by the b.v.p. (1.1.1)-(1.1.2). Then theorem 1.1.2 reduces the equiconvergence problem for operator L to the analogous one for its square. Moreover, the α-numbers for L^2 possess a remarkable property: $\alpha_{tk}(L^2)=0$ *if the indices t and k are of different parity.*

Set $n=2m$ and (see, (1.1.5) and 1.2.7))

$$\delta(L) := \exp(\chi/m)\Omega(L) := \frac{\theta(b^1,b^0,L)}{\theta(b^0,b^1,L)}\cdot\frac{1}{\delta(L)^{q+1}}. \tag{7.1}$$

Then the four most important α-numbers may be written as follows:

$$\alpha_{00}(L^2)=\frac{1}{2\pi}\delta(L)\cdot\Omega(L), \qquad \alpha_{m-1,m-1}(L^2)=\frac{1}{2\pi}\varepsilon_q\cdot\Omega(L),$$
$$\alpha_{mm}(L^2)=\frac{1}{2\pi}\Omega(L)^{-1}, \qquad \alpha_{n-1,n-1}(L^2)=\frac{1}{2\pi}\cdot\varepsilon_{m-\frac{1}{2}}\cdot\delta(L)^{-1}\cdot\Omega(L)^{-1}.$$

Further, results of the section 5 combined with theorems from our article [91] yield

Theorem 7.1. *1.Let $L \in (R)$ be an mth order differential operator and f satisfy (5.1). Then (5.4) is valid if and only if (1.14) is fulfilled.*

2.Let $L_1, L_2 \in (R)$ be two mth order differential operators and f satisfy (1.1). Then (6.7) is valid if and only if (1.14) is fulfilled.

In this section we improved formulation of the corresponding statements in [91] and checked several misprints there.

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